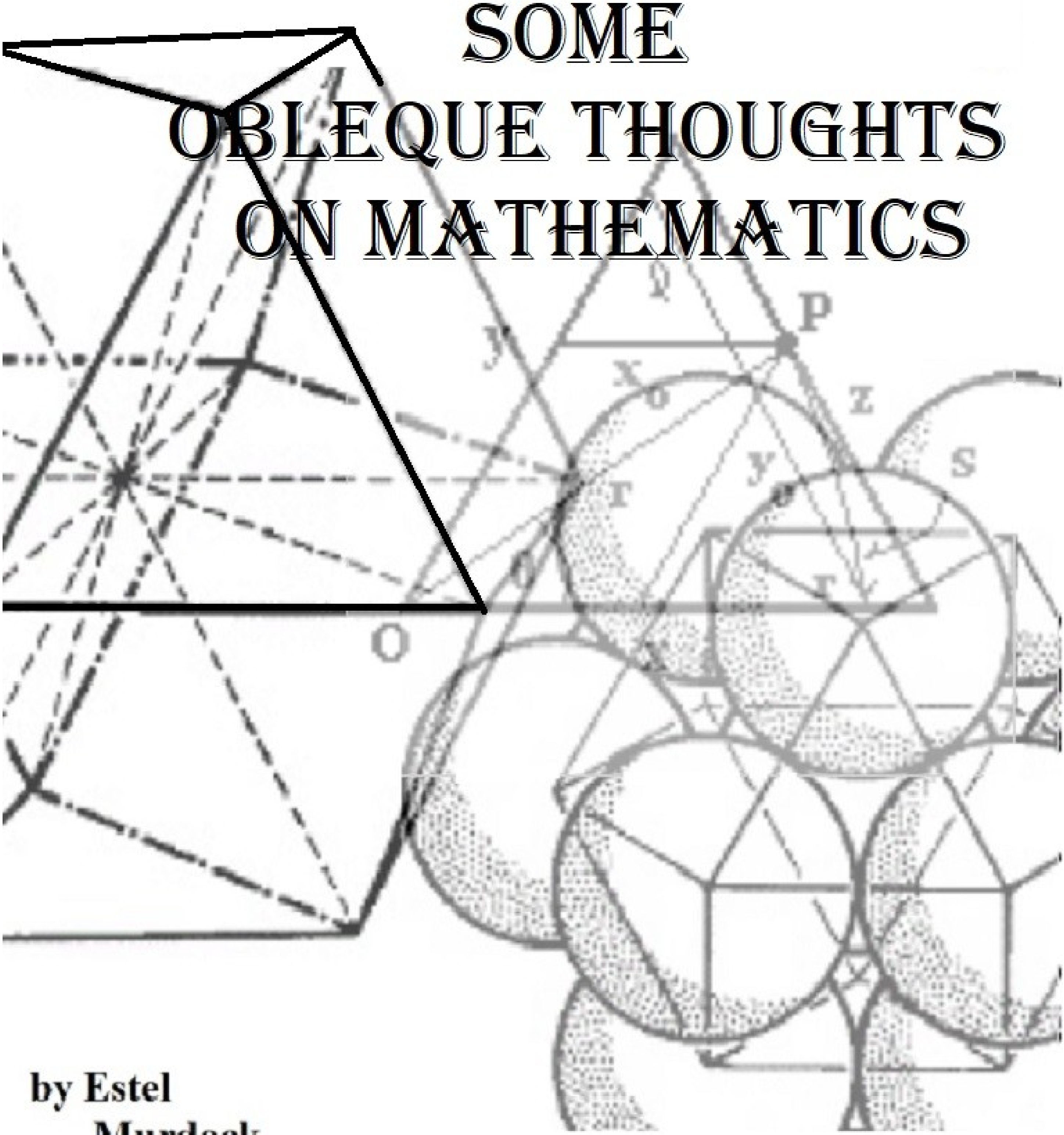
SOME OBLEQUE THOUGHTS ON MATHEMATICS



by Estel Murdock

inspired by Buckminister Fuller

Estel Bradley Murdock
162 S. 500 E. #204
Provo, Utah 84606

Some Oblique Thoughts on Mathematics

by Estel B. Murdock

Preface

This work is in progress. It is never finished. I add to it year by year, tear it apart and put it back together. There may be a lot of mistakes. I am the only one I know who knows what I am doing, and therefore am the only editor. If anyone reads this, feel free to criticize it, correct it, reorganize it, whatever, and write to me at konki47@yahoo.com.

I do not consider myself a mathematician nor a geometrician. I read the works of Buckminister Fuller and found that he had already explained some of my first ideas about triangles, and so he held my interest. I have always wanted to simplify mathematics, and when he came up with the volume of a sphere as being $5r^3$ tetrahedrons instead of $4/3\pi r^3$ cubes, I was hooked. If anything makes math simple in an organized manner I am all for it.

This book is based upon his ideas. If it reads non-professional, I admit that I am only a hobbyist. This book should be a blog on the Internet. It may not be a serious work, but it is full of ideas that could change the way we look at things. It is simply some notes that I have made on the things I have thought about.

I was once told by a math professor friend of mine that he did not see anyone else doing a serious work basing math on geometry, specifically the 60° triangle. The closest I have come to seeing something that resembles my work is the hexagonal coordinate system people are using to design backdrops for computer games. I have read that mathematics is no longer based on geometry. Some college professors even claim that it never was founded on geometry, and that the two fields have always been separate. Buckminister Fuller disagreed. He said that it was, and that there was a primal mistake in this foundation. It was this, that the ancient geometers or mathematicians based their measurements on the length of the diagonal of the unit square as $\sqrt{2}$. This makes things complicated from the beginning. Buckminister Fuller corrected this by labeling the diagonal of the unit square as simply 1. Then the length of each side becomes $1/\sqrt{2}$.

Extending this to a cube, the hexagonal coordinate system is inside the cube as a tilted hexagon and is at 45° from the central vertical line on the side of the cube. The hexagonal coordinate system does all the simplification.

Sometimes it takes a non-professional to see things others cannot see. Computer programmers using the Manhattan distance diagonal may have discovered something very useful. Mathematicians also use the Manhattan distance, but both groups of people just skirt the ideas that I bring up in this book. Everything degenerates into a triangle, and all triangles have a one-to-one correspondence with the equilateral equiangular triangle.

Introduction

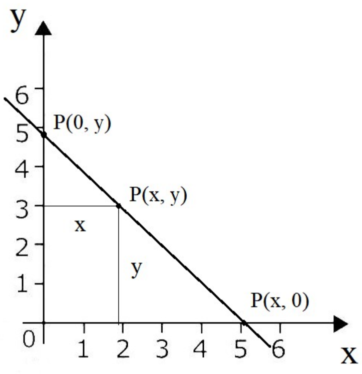
The Pythagorean Theorem Abandoned

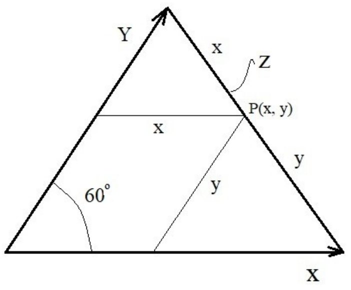
Labels

Before I say anything, let me give a few labels. I can understand what I'm talking about, but it has been hard in the first edition to tell the difference between a line segment and its length. In this edition I have separated these two concepts.

I will call *line segments* with a label of a, b, c, … etc.
lengths of lines segments or coordinates as x, y, z, … etc.
major lines, such as the side of a triangle, may be designated with *capitals;*
vectors will be shown as **a, b, c**, … etc., and (also known as abstract vectors)
lengths of vectors as … **x, y, z,** … etc.
points will be capital letters such as P or P(x, y), etc.
(AP → a) means AP is called a.
Whenever | | is used, it is the absolute value of a number, not distance.

To Start With:



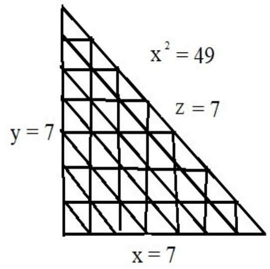


Draw a line L from the y-axis to the x-axis in a 90° coordinate system. There are two numbers x and y less than the intersections, P(x, 0) and P(0, y) of either axis, such that x and y become the coordinates of any point P on line L. Now when the y-axis is rotated down 30°, x + y becomes the length **z** of the line L. This creates an equilateral, equiangular triangle.

Theorem: Given any line L, and any point P on L which divides L into two line segments a and b, then the lengths x and y, respectively, of these two line segments add to the length z of line L. In other words, z = x + y. This is because any line extending from the x-axis or the y-axis to the line L, which we shall call the z-axis, when the three axes form an equilateral triangle, form inner equilateral triangles, and all sides being equal, the coordinates, x and y, of the point P are transferred to line L, so the length of L is x + y = z. In other words, the two segments a and b of L have lengths x and y, respectively, such that x + y = z.

That does away with the Pythagorean Theorem, that is, if you use a 60° coordinate system which this book is all about. You will find that not only do we do away with the Pythagorean Theorem, but you may as well get rid of π also and other complications. Even the trigonometric functions become simpler when dealing with only 60°.

Looking back at the right triangle, each side can be divided into n line segments. That number n is the **meta-length** and is the same for each side. Connecting each point of division of one side to each point of division from each opposite side (there are two opposite sides to each side in a triangle), then the resulting grid on the inside of the triangle consists of similar triangles to the outer triangle. These triangles measure the area inside the outer triangle.



If the sides of the triangle are a, b, and c, then the lengths of those sides are, respectively, x, y, and z. If each length is divided into n sub-lengths such that the number of sub-lengths on one side is equal to the number of the sub-lengths on of the other two sides, then that number squared, n^2, is called the **meta-area**. It is the number of similar triangles inside the triangle.

Divide each side of any triangle into an equal number of divisions with the above definition, use those divisions to draw lines parallel to each of the sides so they intersect within the triangle, and these intersections create planar divisions within the triangle. These divisions are similar-looking triangles, each one having the same area. We can say that each smaller triangle has a unit area. If the number of divisions (line segments) of one side of the larger triangle is n, the triangle whose inner space has been divided, then the number of inner triangles is n^2 and is called the meta-area. It is not an area, but the number of divisions of the triangle.

With this in mind, look at the equilateral triangle. Any triangle being measured with meta-lengths and meta-area is equivalent to an equilateral triangle and can be treated as one. Also, using this method of measurement, you are using pure number instead of things like inches or millimeters, etc.

Some Considerations

Postulate: Any line extending from one side of an equilateral triangle to the opposite side and parallel to the third side forms another equilateral triangle.

Corollary: rotating a line l does not change the length of l.

Definition: the lengths x and y of a and b on Z are one dimensional coordinates, while the lengths x and y of l_x (a line segment of length x) and l_y (a line segment of length y) within the XYZ triangle are two dimensional coordinates.

Theorem: given any line L, if P divides L into two line segments a and b, then the length of L is z = a + b.

Definition: Having formed two equilateral triangles within the XYZ triangle, the space left over is a parallelogram yxyx. Its area is xy, comparable to a rectangle within a 90° coordinate system.

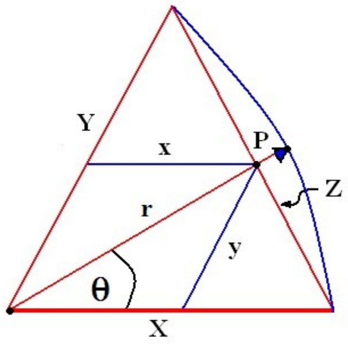
Theorem: If a point P divides Z into two line segments a and b such that their lengths are x and y, respectively, and if P is the point on Z that joins the triangles abc and a'b'c', where c and c' also have lengths x and y, then z, the length of Z, is equal to x + y.

A Generalization of z = x + y

It has been shown that any line segment Z has a length z, such that z = x + y. It can be clearly seen that

$$z = (0 + x) + (0 + y).$$

This represents any line, such as d which has been rotated with its origin attached to the intersection of X and Y and is equal in length to any side of the equilateral triangle XYZ. The endpoint of d describes an arc as it rotates from the endpoint of X to the endpoint of Y, the length z of Z. It rotates from 0° to 60°, which is 1/6th of the circle inscribing a hexagon. As d is rotated, it cuts across the line Z, which is opposite the XY intersection or origin of the hexagon. At the intersection of d and Z is the point P which divides Z into two line segments, a and b, with lengths x and y, respectively. Let w be the length of d. Then w = x = y = z in length, the length w = x + y as well as z. (I am using a convention of enumerating the sides of the triangle and not the apexes to name the triangle.)



Theorem: Let d, with a length w, be the line X rotated from 0° to 60° and from X to Y, which is the length z of Z. As d intersects Z, Z is divided into two line segments a and b such that the length of a is x and the length of b is y. Since d = X = Y = Z and the length z = x + y, then the length w = z.

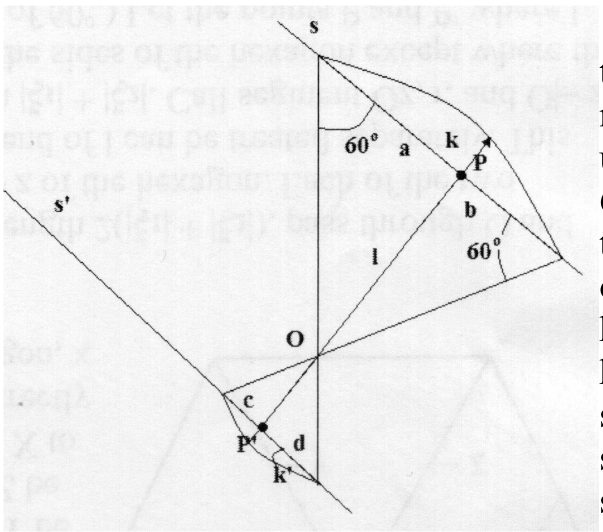
Theorem: If w = z = x + y then z = w = (0 + x) + (0 + y).

This shows that for any two line segments s and r, rotating around the origin of a hexagon, the length of s – r, which is $w = (x_2 - x_1) + (y_2 - y_1)$.

Thus, we have done away with the Pythagorean Theorem as a major way of measuring length. This takes care of any line segment within the hexagonal plane by the construction of an equilateral triangle (1/6th of a hexagon) anywhere within that plane.

As l Passes Through Two Parallel Lines

It can be generalized that the length i of any line segment l having a rotation within 360° is equal to the sum of the coordinates of two points P and P'. It is sufficient to show that i is the sum of two sets of coordinates as l intersects two parallel lines. This can be accomplished by having l rotate about the center of a hexagon while intersecting any two opposing sides.



Let line l intersect two parallel lines s and s' such that both angles of intersection are 60°. There is a midpoint O on l between the two intersections. Rotate l until the next angles of intersection on s and s' reach 60°. This produces line segments k on s and k' on s'. Let the points of intersection be designated as P on k and P' on k'. It has been shown that where l has sliced through k there are two line segments a and b created such that $k = a + b$. Now the same situation exists for k', l simultaneously slicing through k' creating two line segments d and c such that $k' = d + c$. Now, it has been shown that because we have two angles on line k of 60°, then if there are two lines a' and b' extending from these two angles, the angles of intersection of a' and y' is also 60°. That creates an equilateral triangle, and if $k = a'$, with sides X, Y, and Z. Likewise, we have another equilateral triangle X', Y', Z' where $k' = Z'$. The apexes of these two lines kiss at O. Now the line segment OP that spans k has the same length as X, Y, or Z, and the line segment OP' has the same length as X', Y', or Z'. Therefore, the lengths of OP and OP' are the same lengths of k and k' respectively. So, adding, $k + k' = OP + OP' = l = (a + b) + (d + e)$. If the length of l is m, and the lengths of a, b, c, and d are u, v, u', and v' respectively, we have

$$m = (u + v) + (u' + v')$$

where the lengths are of any combination of $-/+\ x$, $-/+\ y$, or $-/+\ z$ to denote any two opposing sides of the hexagon (with axes X, Y, and Z).

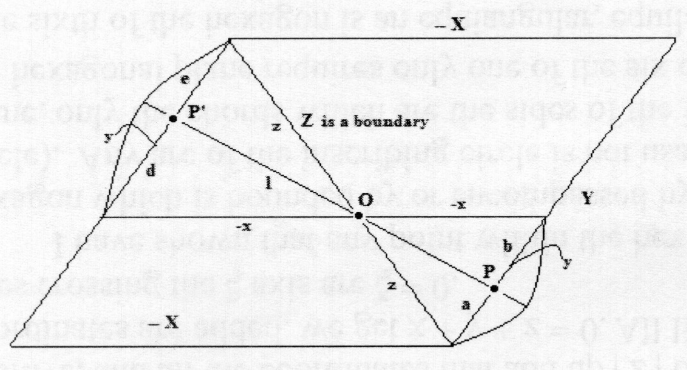
Generally speaking, any line l has a length z such that $z = x + y$.

The Pythagorean Theorem is not needed to find the length of a line. Numbers can be left rational, but it takes a coordinate system composed of the axes of the hexagon.

Corollary: The length z of any line segment, no matter at what angle, is equal to $x + y$.

As a Line Passes Through a Boundary Z

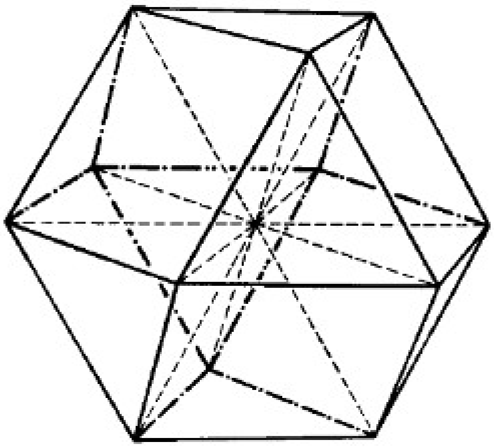
The hexagon is made up of 6 equilateral triangles with boundaries, the axes X, Y, and Z. Choose any line l that passes through a boundary Z at a point O on Z. Place the endpoints of l anywhere within two adjacent triangles (–X)YZ and Z(–X)Y. Let the line l pass through points P on the line segment f and P' on the parallel line segment f '. The coordinates of a point P are a and b such that $f = a + b$. The coordinates of the



point P' are d and e such that $f' = d + e$. From P a extends out to the junction of Z and $-X$, where b extends from P out to $-X_c$, between an upper $-X$ and a lower $-X$. From P' d extends to $-X_c$, and e extends to the junction of $-X_{upper}$ and Z. Rotating l about the center O from $-X_c$ to Z as it crosses over f and f ' shows that the length of l at 0° and 60° is equal to the length of $-X$ and thus to any of the sides of the 6 equilateral triangles of the hexagon. Therefore the length of l is always equal to the length of $-X$. The chords f and f ' of the two arcs thus produced by the rotation of l completes two smaller equilateral triangles as lengths $(-x)yz$ within the larger triangle $(-X)YZ$ and lengths $z'(-x')y'$ within the larger triangle $Z(-X)Y$ with their apexes at O. That part of l that extends from O out to the arc between $-x'$ and z' can be rotated as a radius r 60° about O. This radius r is the same length as $-x$, y, or z, and thus, $r = a + b$. As $r = a + b$, $r' = d + e$, and as $l = r + r'$, $l = (a + b) + (d + e)$.

This equation can be generalized as $l = m + n$, the sum of the coordinates of the two points P(a, b) and P'(d, e). This definition of the length of a line holds for any line within any sized hexagon, called a 60° coordinate system. The Pythagorean Theorem used in the 90° coordinate system is not needed, and this new definition of the length of a line is a lot simpler.

A Line Passing Through the Vector Equilibrium



There are four hexagons within a VE. They all intersect at one point at the center of the VE. Let there be a line l running from one apex through the center point and to the opposite apex across the equatorial hexagon. Because of the structure of the VE, there is one more hexagon passing through that line. This happens through every one of the three axes passing through the equatorial hexagon. This means that for any line coincident with an axes in any hexagon in the VE, there are two hexagons passing through it. There are 4 hexagons x 3 axes each = 12 configurations of two hexagons intersecting to define a line. But 6 of these configurations are repeats. There is 1 of 3, 2 of 3, and 3 of 3 that are repeats. So the number of configurations are

$$3 + (3 - 1) + (3 - 2) + (3 - 3) = 6.$$

The intersection of any two hexagons within the VE define a line (also an axis) passing through both hexagons, and there are six of these configurations. Each of these lines can be rotated as the VE is rotated to cover any line passing through the center point of the VE. There are six other lines to speak of passing through three sets of two apex touching tetrahedrons and three sets of two apex touching pyramids. The apexes of both tetrahedrons and pyramids are the central point of the VE, so the six lines passing through either the pyramids or the tetrahedrons pass through the central point of the VE. That is twelve lines passing through the central point.

Any line passing through the central point is also defined by four numbers $l(a, b, c, \theta)$.

The first three numbers are the closest orthogonal distances to any x, y, or z axes, and the angle θ is the angle of rotation of one of the hexagons the line passes through. This is the same for lines passing through the central point and a pair of tetrahedrons with angle θ the rotational distance from the leftmost hexagon. The numbers a, b, c have optional values as orthogonal distances from the edges of the outer triangles, that is, using square angles. Any line passing through the central point and two of the pyramids has numbers $l(a, b, c, d, \theta)$ such that the a, b, c, and d are the orthogonal distances from the sides of the squares, using square angles. The angle θ is the rotational distance from the leftmost hexagon.

Line not Passing Through the Center of the VE

Any line not passing through the central point passes through at most three hexagons, creating three points of the form $P(x, y, z)$, each one of which, having coordinates x, y, and z. These coordinates are three points P_x, P_y, and P_z, one on each side of an equilateral triangle located on one of the three hexagonal planes. So there are nine coordinates on three planes to deal with, but they diminish because of sharing or counting twice. You wind up with six coordinates as previously stated, talking about the number of configurations: x_1, x_2, x_3, y_1, y_2, z_1. One point would be in the space of the X-plane (horizontal) with x_1, x_2, x_3, and two Y-planes with y_1 and y_2, each one sharing z_1. Another space could have the same coordinates except for the z coordinate which would be $-z_1$. There are again, 6 of these spaces for points to be in, three positive, and three negative, meaning that the z-axis is either positive or negative.

Just as the length of a line in a plane is $l = (u + v) + (x + y)$, the addition of two lines, the line within the VE would be described as $l_{VE} = (r + s + t) + (u + v + w) + (x + y + z)$, the addition or intersection of three planes.

The different combinations are: $(x_1, y_1, z_1), (x_3, y_2, z_1), (x_1, x_2, x_3)$
$(x_1, y_1, -z_1), (x_3, y_2, -z_1), (x_1, x_2, x_3)$
$(x_4, y_4, z_4), (x_6, y_5, z_4), (x_4, x_5, x_6)$
$(x_4, y_4, -z_4), (x_6, y_5, -z_4), (x_4, x_5, x_6)$
$(x_1, y_1, z_1), (x_3, y_2, z_1), (x_7, x_8, x_9)$
$(x_1, y_1, -z_1), (x_3, y_2, -z_1), (x_1, x_2, x_3)$

For example, taking the first combination,

$$l_{VE} = (x_1 + y_1 + z_1) + (x_3 + y_2 + z_1) + (x_1 + x_2 + x_3)$$

from points $P_1(x_1, y_1, z_1)$, $P_2(x_3, y_2, z_1)$, $P_3(x_1, x_2, x_3)$, where each of these points lie on an equilateral triangle within one of the three hexagonal planes. The fourth hexagonal plane has the line passing only through the outer perimeter. (That could be a fourth coordinate which can be overlooked at present.)

The line l_{VE} penetrates the three hexagonal planes at the primary points P_x, P_y, and P_z. Each of these points have three coordinates, which, themselves, are points, one on each of the sides of the equilateral triangle. There is a line passing across that secondary point to an opposite apex, thus dividing that side into two parts. These secondary points have coordinates, the lengths of the divided sides of the equilateral triangles.

So, we start with an equilateral triangle with three axes coming from each of the three apexes crossing each other, creating a point P_i, as they extend across to the opposite side from the apexes. It is noted that these axes are the same in length as the sides of the triangle. Where they cross the sides of the equilateral triangle they create three other points P_x, P_y, and P_z, which become the coordinates of the primary point P_i. Now, each of the points P_x, P_y, and P_z have coordinates, the lengths of the line segments created by each division of the sides by the axes. Each equilateral triangle is a segment of one of the intersecting hexagonal planes.

Thus, we have a line l_{VE} passing through a hexagonal plane at point $P(P_x, P_y, P_z)$. P has three coordinates P_x, P_y, and P_z which are points on each side of an equilateral triangle which is the local area that l_{VE} passes through. The line l_x comes from an apex of the triangle and passes through P_x. Lines l_y and l_z also pass through P_y, and P_z. Each of the lines l_x, l_y, and l_z are the same length as the sides of the triangle. For example, a is the bottom side of an equilateral triangle. A copy of a is called l_x and is rotated up to P_z on c where it crosses P_z and divides line l_z into two divisions called e_x and f_y. Thus P_z has coordinates x and y. Likewise, P_x and P_y have coordinates y, z and x, z, respectively. In terms of these coordinates,

$$P(P_x, P_y, P_z) = P([y, z], [x ,z], [x , y]).$$

Each of the axis lines of two opposing equilateral triangles sharing one apex has its counterpart in the other equilateral triangle on the other hexagonal plane. These counterpart lines are said to be parallel. Thus, through each pair of parallel lines passes a plane. There are three such planes produced, and these three planes intersect at and thus define the line l_{VE}.

It has been established that the coordinate $z = x + y$ and $x + y + z = 0$. The third coordinate can be included such that $P(P_x, P_y, P_z) = P([y, z, x], [x, z, y], [x, y, z])$. Thus, between any two hexagonal planes, l_{VE} is defined as $l_{VE} = (a, b, c) \cap (d, e, f)$. Of course, through three hexagonal planes, l_{VE} is defined as $l_{VE} = (a, b, c) \cap (d, e, f) \cap (k, l, m)$. These three groups of line segments are the three intersecting planes that define l_{VE}.

Any one of these points P_1, P_2, and P_3, may have zeros as coordinates. If two of these points have zeros as coordinates, l_{VE} is not defined.

Not being a mathematician, I am also interested in $l_{VE} = (x_1 + x_2 + x_3) + (y_1 + y_2) + z_1$. This seems to be a representation of the state space of the VE. I shall write it this way: $S_{VE} = [x_1 , x_2 , x_3] [y_1 , y_2] [z_1]$. There should be a function F such that $F(S_{VE}) = (a, b, c) \cap (d, e, f) \cap (k, l, m)$, which are all the combinations of the coordinates x_i, y_i, and z_i. Why I included more than one z coordinate is because of the fourth hexagonal plane. I could show how the state space S_{VE} and all its functions define aVE, but that I will leave for someone else or for a later time. I could mention that a VE = ∅, that is, all the vectors in VE add up to zero. It is then understandable that three orthogonal vectors add up to zero, and thus any line passing through VE is an empty space, a void, or has a zero vector. That means that the two endpoints of l_{VE} share the same coordinates. You could think of l_{VE} as a linear hole.

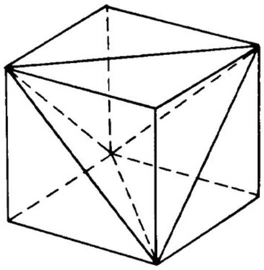
Chapter One

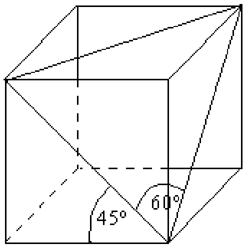
Making Things Rational

The Diagonal of a Square and The Isotropic Vector Matrix

All the world is a cube, or so we thought. It is standard to use the 90° coordinate system in all mathdom. Volume is measured in cubic centimeters. Area is measured in square meters, etc. Whenever scientists or mathematicians sought for space beyond the cube, they invented the hypercube which is n-dimensional.

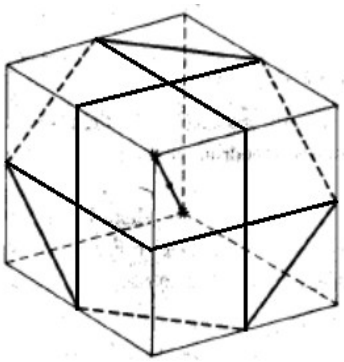
What if they were looking in the wrong direction? What if there is a whole different geometry *inside* the cube? Just as a triangle is the most basic polygon that encloses area instead of a square, that is, it has the smallest number of sides which can define a plane, the tetrahedron is the smallest polyhedron instead of a cube which encloses volume. A tetrahedron has only 4 faces, whereas a cube has 6. You can fit less space into a tetrahedron. In fact, it requires 5 tetrahedrons to make 1 cube.

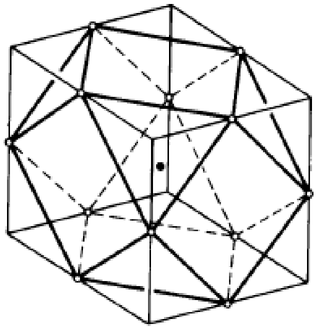


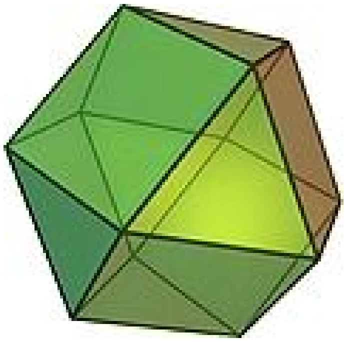


If you take three connecting diagonals on three faces of the cube, you see an equilateral triangular plane inside the cube. Each facial diagonal is 45° from the edge of the cube, and any two connecting diagonals are 60° away from each other on that triangular plane inside the cube. So inside the cube is another geometry altogether 45° away from the normal 90° coordinate system.

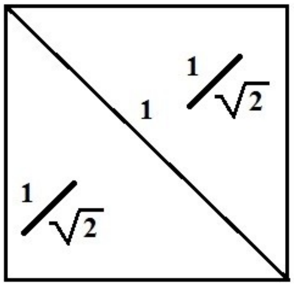
Take the above cube and cut it into 8 smaller cubes. Each face of the cube is cut into four smaller faces. Draw the diagonals of the outside faces of the smaller cubes so they connect at the middle of each edge of the larger cube. These diagonals are automatically connected at the vertexes of four hexagons or a solid by the name of cuboctahedron. It is also like taking eight unit tetrahedrons from the corners of the larger cube. This cuboctahedron is the manifold of a new geometry and a new way of measuring volume and area. It could also prove to be the basis of a new algebra and calculus.







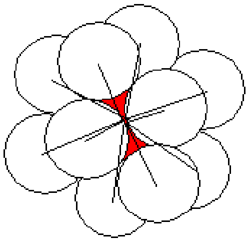
From this inner geometry we can extrapolate a few things. If each edge of the cuboctahedron is considered to be two vectors (the diagonals of the smaller cubes) pointing at each other, one coming from one vertex and the other from the opposite vertex, and both equal, then we call it an isotropic vector matrix or vector equilibrium or shortened to VE.



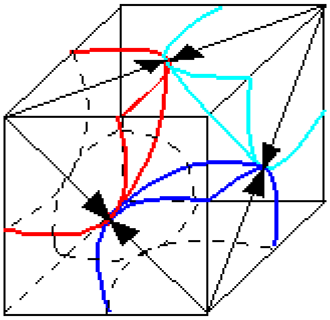
The primary difference between this isotropic vector matrix and the old cubic vector matrix is that the diagonals within a unit square on the faces of a cube are of unit value making the sides of each square $1/\sqrt{2}$ or $\sqrt{2}$ / 2 which is the value of both sin θ and cos θ and are irrational. This leaves the cube with an irrational measure and the measure of this inner geometry rational.

Let me come at this from a different angle.

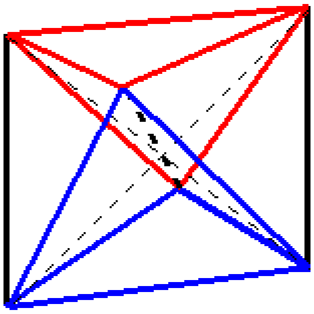
The **Isotropic Vector Matrix** comes from the closest packing of unit radius spheres. Each sphere within the isotropic vector matrix has 12 surrounding spheres. Connecting the centers of each of the 12 spheres to the center of the nuclear sphere are 12 double radii radiating from the nuclear sphere. (One radius from each sphere connected to one radius from the nuclear sphere.) Each axis is separated by 60º from an adjacent axis. This angle of 60º is a property of the adjacency of the spheres.



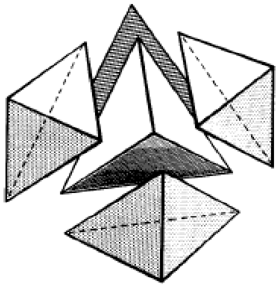
Converting from Irrational to Rational

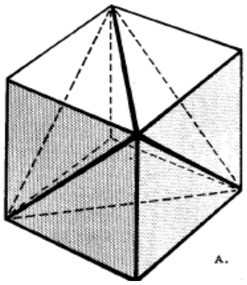


Four $^1/_8$ unit spheres within the isotropic vector matrix come together within the cubic vector matrix such that four of the corners correspond to the centers of the spheres, and the cube's face diagonals coincide with the sphere's radii, making the diagonals of the square sides two units each, that is, two opposing vectors. Therefore, each square side on the cube has a diagonal of 2, and each edge of the cube has a length of $\sqrt{2}$. The area of each one of these squares is therefore $(\sqrt{2})^2 = 2$ traditional unit squares. What I mean by traditional is that the face of a traditional unit square has a face diagonal of $\sqrt{2}$. But for our purposes, a unit square has a unit diagonal with sides of $\sqrt{2}$.



The area of the cube using the irrational sides of $\sqrt{2}$ is $(\sqrt{2})^3 =$ 2.828428. The unit octahedron is made up of four unit tetrahedrons. Therefore, the volume of the octahedron is 4 tetrahedrons. That is the smallest unit of volume. Cut two of these tetrahedrons in half to make four ½ – tetrahedrons, each having a volume of ½. When four ½ – tetrahedrons are added to each face of one unit tetrahedron, the smallest cube is created because $4(^1/_2) + 1 = 3$, an easier way of calculating the smallest cube than using the side of $\sqrt{2}$. This is the 5 tetrahedrons that creates a cube.



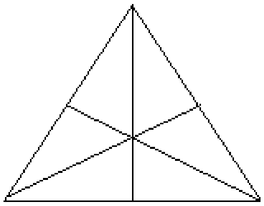


Comparing this rational volume of 3 to the calculated volume using the $\sqrt{2}$, we get the synergetics conversion factor of $^3/_{2.828428}$ = 1.06066. Using this conversion factor conventional areas and volumes are converted to rational areas and volumes. This conversion factor of 1.06066 = $\sqrt{(9/8)}$.

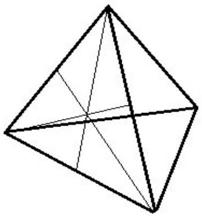
For areas, 2 dimensions, $\sqrt{(9/8)}$ is triangled to become $(\sqrt{(9/8)}\,)^2 = {}^9/_8$.
For volumes, 3 dimensions, $\sqrt{(9/8)}$ is tratrahedroned[1] to become $(\sqrt{(9/8)}\,)^3 =$ 1.193243.
Here are some practical examples.

The Great Pyramid at Giza has a volume of 2.5 million cubic meters. 2.5 x 1.193243 = 3, that is 3 million tetrahedrons.	The Chalula Pyramid in Mexico has a volume of 4.45 million cubic meters. 4.45 x 1.193243 = 5 million tetrahedrons.

Numbers become rational within this inner geometry. This is due to making the diagonal within each square the unit instead of the mistake of making the side of the square the unit.



Volume as the Tetrahedral Part

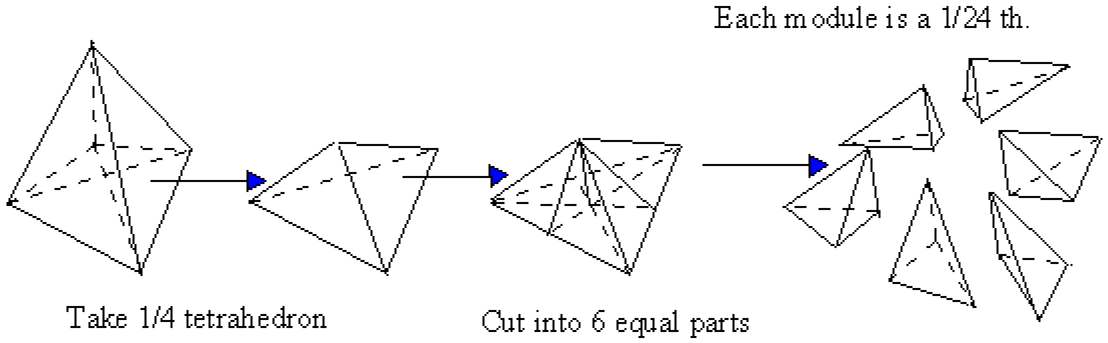


Within the cuboctahedron is a plane, a hexagon, made up of six equilateral, equiangular triangles, having three axes from each corner to the opposite side dividing the triangle into 6 right triangles. On each face of a unit

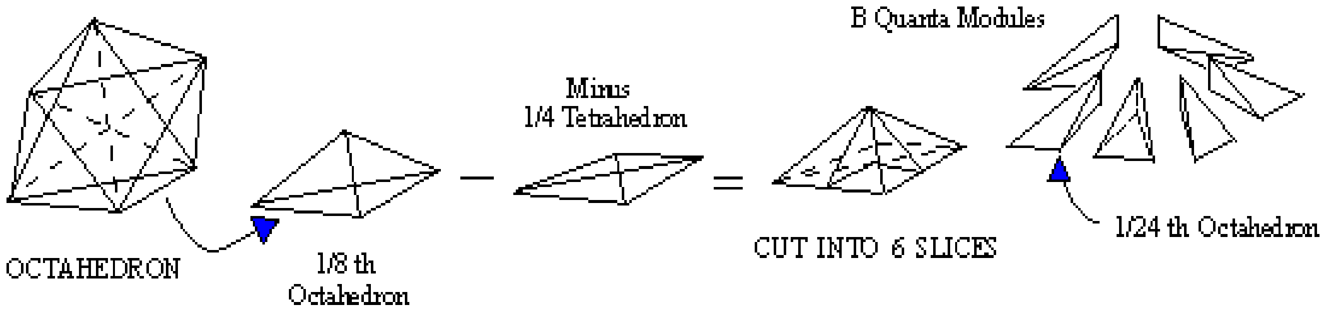
---

[1] Triangling and tetrahedroning will be explained later, being equivalent to squaring and cubing.

tetrahedron, these axes stretch out to become planes within the tetrahedron, and they divide the tetrahedron into 6 right triangles times 4 sides = 24 modules, each one being called an A–Quanta Module.

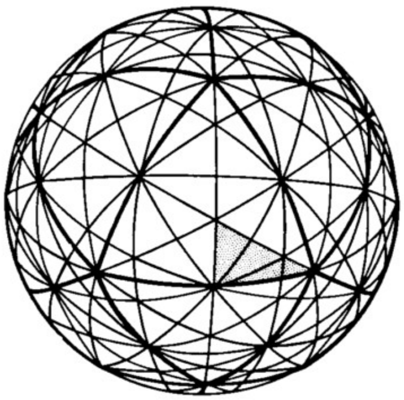


A whole octahedron has a volume of 4 unit tetrahedrons. 1/8th of that octahedron is a ½ tetrahedron**.** Taking away ½ of that gives you a quarter volume. Dividing that quarter volume by 6 gives you a 1/24th volume. Call that the B Quanta Module. The A and B Quanta Modules are equal in volume, but the A Quanta is part of a tetrahedron, and the B Quanta is part of an octahedron.



Neither the tetrahedron nor the octahedron are all space fillers. It takes both to fill all of space. That is why to describe any part of space, you need a collection of A and B quanta modules.

Volume of a Sphere



For a unit sphere, the radius being one, the conventional volume is $4/3 \pi r^3 = 4.188790$ cubes. Multiplying by the synergetics constant for 3 dimensions, $[\sqrt{(9/8)}]^3$, that is, 4.188790 x 1.193243, we get 4.998425, that is, 5 tetrahedrons. It has therefore become rational and the new formula for spherical volume should be **V =** $5r^3$ tetrahedrons.

An icosahedron has 20 equilateral triangles on its surface. Projecting the icosahedron onto the surface of a surrounding sphere and taking each of the 20 now spherical equilateral triangles and dividing them into 6 right triangles, you get 20 x 6 = 120 triangles. Extending them to the center to form 120 tetrahedrons, each has the volume of an A or B Quanta Module, or a volume of 1/24th of a unit tetrahedron. Therefore, 120 x $^1/_{24}$ = 5, the volume of the unit sphere. This is easier than using the conventional formula for spherical volume.

Using the Volume of a Cube to Find the Volume of a Sphere

The volume of the unit cube is 3. Taking away the 8 corners, such that each is $^1/_{16}$ th of a unit tetrahedron, produces the Vector Equilibrium. 8 x $^1/_{16}$ = ½, so the sum of the corners taken away is ½ of a unit tetrahedron, showing that the volume of a unit or basic Vector Equilibrium is 3 – ½ = 2 ½ . Therefore, the volume of a unit sphere, being 5, is the same as the volume of two Vector Equilibriums and has the same volume as 120 A and B Quanta Modules.

More Volumes

From the examples thus given, it is seen that using geometry, that is, the inner geometry found inside the cube and 45° away from it, to find area and volume is simpler than using conventional means. This new method gives rational solutions.

All symmetric forms can be measured simply using the A and B Quanta Modules as the unit of measure. This is without the use of π.

This following chart shows some examples of the volumes of some solids based on the volume of the tetrahedron as unity.

SYMMETRICAL FORM (based upon the closest packing of unit radius spheres)	***TETRA VOLUMES*** (the unit of volume being one unit tetrahedron)	***A and B QUANTA MODULES*** (multiples of 12 spheres surrounding a nuclear sphere)
Tetrahedron	1	24 = 2 x12
Vector Equilibrium	2 ½	60 = 5 x12
Cube	3	72 = 6 x12
Octahedron	4	96 = 8 x12
Sphere	5	120 = 10 x12
Rhombic Dodecahedron	6	144 = 12 x12

All of the above symmetrical forms are of the form Nr^3 where N is the number of tetrahedrons in the volume of the basic form and r is the frequency of that form. For example, using r as equal to 2nd frequency, $r^3 = 2^3 = 8$ (frequency is expansion of time through space)

SYMMETRICAL FORM Nr^3	***TETRA VOLUMES*** r^3 (r = 2)	***A and B*** ***QUANTA MODULES***
tetrahedron	8	08 x 24 = 192 = 8 x 24
vector equilibrium	20	20 x 24 = 480 = 8 x 60
cube	24	24 x 24 = 576 = 8 x 72
octahedron	32	32 x 24 = 768 = 8 x 96
nuclear sphere	40	40 x 24 = 960 = 8 x 120
rhombic dodecahedron	48	48 x 24 = 1152 = 8 x 144

Remember that an A or B quanta module is 1/24th a volume.

The number of A and B quanta modules are shown here as multiples of tetrahedrons of frequency 2 times the number of A and B quanta modules in their primary forms.

The number of A and B quanta modules are also the same number as important angles with a system of angles, lines and planes making up three dimensional forms.

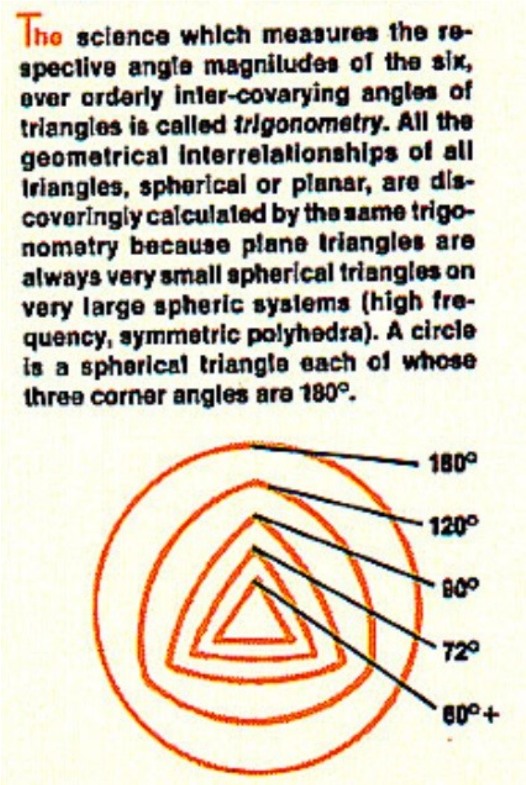
Taking the idea of 12 unit spheres surrounding a nuclear sphere, if we had another layer of unit spheres, and another layer of unit spheres, we come up with the formula of $10R^2$ +2. We obtain from this the list of numbers: [2, 12, 42, 92, …].

Polyhedron:	Number of vertexes:	Number of Angles about a Vertex:	Angles about a Vertex:	Sum of Angles around each Vertex:	Sum of Angles within the Polyhedron:
Triangle	3	2 x	60º	120º	60º x 3 = 180º
Tetrahedron	4	3 x	60º	180º	180º x 4 =720º
Octahedron	6	4 x	60º	240º	240º x 6 = 1440º
Cube	8	3 x	90º	270º	270º x 8 = 2160º
Icosahedron	12	5 x	60º	300º	300º x 12 = 3600º
Dodecahedron	20	3 x	108º	324º	324º x 20 = 6480º
Vector Equilibrium	12	2 x 2 x	90º 60º	180º + 120º = 300º	300º x 12 = 3600º

There is a beautiful formula for *d*-dimensional convex polytopes *P*, called *Brianchon-Euler-Gram theorem* (see for instance Theorem 22 here) which generalizes the classical formula for the angle sum of polygons:

$$\sum i=0d(-1)i\sum F,\ dim(F)=i\angle F(P)=0.$$

Here $\angle F(P)$ is the solid angle of P at the face F (of dimension i). *by Moishe Cohen on Yahoo! Answers.*

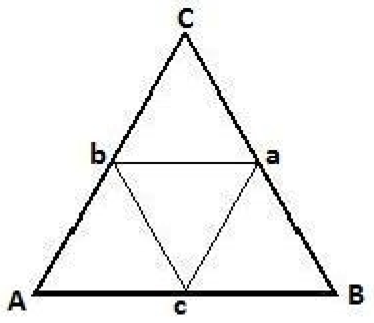


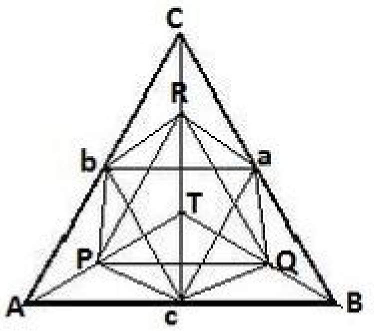
The science which measures the respective angle magnitudes of the six, ever orderly inter-covarying angles of triangles is called *trigonometry*. All the geometrical interrelationships of all triangles, spherical or planar, are discoveringly calculated by the same trigonometry because plane triangles are always very small spherical triangles on very large spheric systems (high frequency, symmetric polyhedra). A circle is a spherical triangle each of whose three corner angles are 180°.

Spherical Angles from Buckminister Fuller: Expanding from 60°: 72° → 120° → 180°. A spherical triangle with inside angles of 180° is a circle.

The Volume of a Tetrahedron

Each side of a tetrahedron is a triangle which has the properties of an equilateral triangle. So if we divide each side of each of the four triangle faces in half, i.e., half each edge of the tetrahedron and then connect each of the division points to the corresponding points on the other to sides of the triangle, on each triangular face,



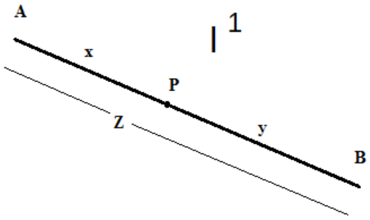


you have a tetrahedron with regular tetrahedrons at each corner with the empty space inside filled with an octahedron. The octahedron in turn is made up of four tetrahedrons of the same volume as the corner tetrahedrons. If these tetrahedrons are named unit tetrahedrons, then there are eight tetrahedrons as the smallest division of the outer tetrahedron.

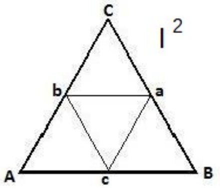
If we apply the fact that any triangle can be treated as an equilateral triangle, then the faces of a tetrahedron can also be treated as equilateral triangles, and therefore, any tetrahedron can be treated as a regular tetrahedron. This is done by dividing the edges of the tetrahedron by the same even number of divisions. That will make the edges of the inner octahedron have the same number divisions as the edges of one of the corner tetrahedrons. As all the triangles have a volume divisible by 4, each volume of a tetrahedron has a volume divided by 8.

Geometrical Progression

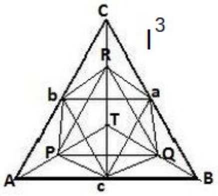
Take a line of length l, triangle it as l^2, and you have the area of a triangle, divisible by 4. Now, tetrahedron it, and becomes l^3, the volume of a tetrahedron.



=>



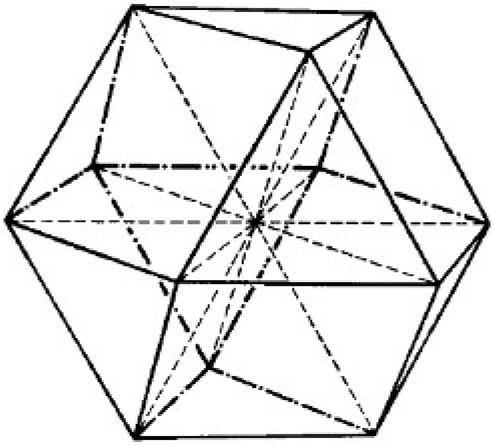
==>



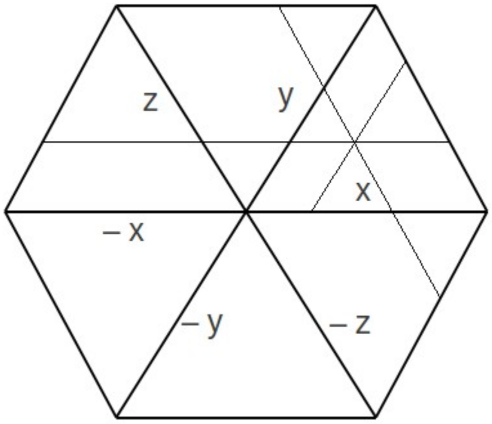
Chapter Two

The Hexagonal Plane

The Hexagonal Plane



The Vector Equilibrium is a 4–dimensional manifold. It exists 45° from an enclosing cube. It can be sliced four ways to produce 4 different planes and has 4 axes, each axis being coplanar with each of the 4 planes. Each one of these planes is a hexagon. The hexagon has three axes, each one drawn from corner to opposite corner, all three meeting in the center and forming six equiangular equilateral triangles. Each one of these hexagons in the Vector Equilibrium is a 3–dimensional manifold projected onto a 2–dimensional plane. In other words, a cube projected at 45° onto a surface.



The hexagon represents a flattened cube with its X, Y, and Z axes. (Think of a cube being drawn on a blackboard.) Each point within the hexagon is P(x, y, z) and is found by extending a 60° orthogonal leg from each of the three axes. First, draw a parallel line for each of the X, Y, and Z axes, namely, axes x, y, and z, naming them for the coordinates of a point they represent. Where all of these axes cross you have the point P(x, y, z). The axes x, y, and z parallel the axes X, Y, and Z of the hexagon and generalize the concept of the hexagonal plane, creating six other phantom sextants which reflect the sextants of the original hexagon. The points in each of the original sextants are

P(x, y, z),

P(-x, y, z),

P(-x, -y, z),

P(-x, -y, -z),

P(x, -y, -z),

P(x, -y, -z).

This is according to a counterclockwise rotation. The generalized point is P(α, β, γ). Each sextant is an equilateral triangle, and only two coordinates within an equilateral triangle are needed to determine a point on a plane. Choosing any α, β, γ sextant-equilateral triangle, let a line r extend from the origin of the hexagon to the outer side γ of the triangle. This cuts the outer side γ into 2 segments, α and β which are the sides of two inner equilateral triangles, the α and the β triangle, which share sides with the intervening parallelogram. The upper and right sides of this parallelogram are the α and β coordinates of the point P(α, β) where r crosses the γ line.

Any point within the hexagonal plane is found on one of the sides of a hexagon which is bounded by or encompassed by a circle (sharing its corner points with a circle). Any arc of the inscribing circle is not used in determining a point in the hexagonal plane, only the chords which are the sides of the inscribed hexagon. Therefore, any point within the hexagonal plane requires only the six chords bounded by a circle and not the arcs of the circle. One sixth of the hexagon is the 60° equiangular, equilateral triangle we want to deal with in this chapter. The coordinates of the point P are located on the outer side of each sextant of the hexagon, that is, on the outer side of each equilateral triangle as a vector r from the center of the hexagon and equal in length as the side of the sextant/equilateral triangle or the radius of the circle encasing the hexagon, cuts the outer side into two line segments, a and b such that a + b = c, and the lengths of a, b, and c are respectively x, y, and z. Any point can be expanded as above to include the third coordinate, thus, P(x, y) → P(x, y, z).

More on Coordinates

Because all points on the hexagonal plane are also points within the Vector Equilibrium, in which all vectors added together sum to zero, the sum of all coordinates are equal to zero. Therefore, $\underline{+}x \ \underline{+}y \ \underline{+}z = 0$. In other words, $x + y + z = 0$.

The x, y, and z lines parallel to the X, Y, and Z axes will be referred to as the general axes of the hexagonal plane. So,

the x axis is where $y = z = 0$
the y axis is where $x = z = 0$
the z axis is where $x = y = 0$

As each axis coordinate becomes zero, we have coordinates:

P(x, y, 0)

P(0, y, z)

P(– x, 0, z)

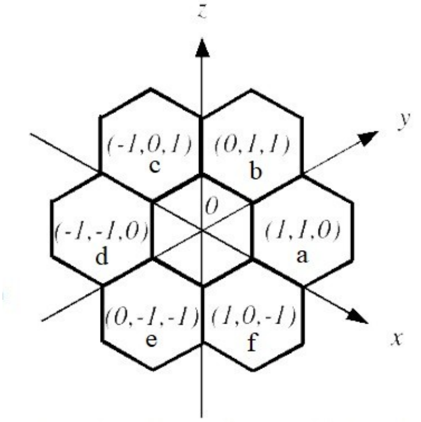
P(– x, – y, 0)

P(0, – y, – z)

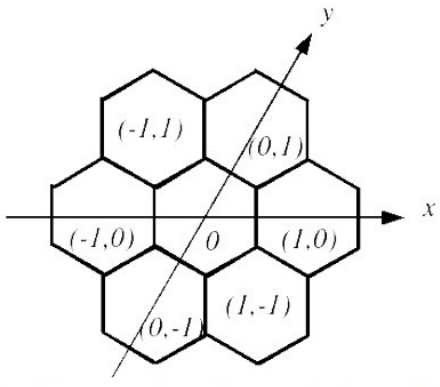
P(x, 0, – z)

Coordinate Vectors of the Hexagonal Tiles*

Each point within the hexagonal plane can be treated as vectors. They work just like the point vectors within the Cartesian or 90° coordinate system. Referring to the points listed above, let us turn each point into a unit vector originating at the origin of the hexagonal plane. So the list of unit vectors counterclockwise are (eliminating each outer coordinate in each sextant):



Example of using 3-element coordinate system to index hexagonal pixels



Example of using a skewed-axis coordinate system to index hexagonal pixels

(You can see the vectors for three dimensions or two dimensions.)

$\mathbf{a} = [1, 1, 0]$,

$\mathbf{b} = [0, 1, 1]$,

$\mathbf{c} = [-1, 0, 1]$,

$\mathbf{d} = [-1, -1, 0]$,

$\mathbf{e} = [0, -1, -1]$, and

$\mathbf{f} = [1, 0, -1]$

Starting with a vector **v** and translating it a distance of

$$\begin{aligned}\mathbf{v} + 5\mathbf{c} &= [x, y, z] + 5\,[-1, 0, 1]\\ &= [x, y, z] + [-5, 0, 5]\\ &= [-5x, 0, 5y]\end{aligned}$$

To generalize, we can always find integers m and n such that, for any two vectors **v** and **u** and two directions (unit vectors) **i** and **j**,

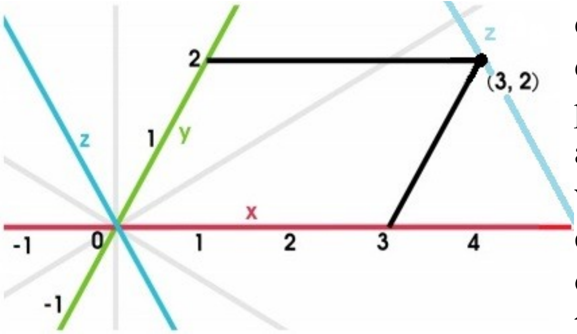
$$\mathbf{u} + m\mathbf{i} + n\mathbf{j} = \mathbf{v} \quad \text{which changes to}$$

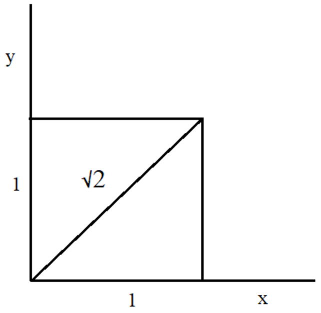
$$m\mathbf{i} + n\mathbf{j} = \mathbf{v} - \mathbf{u}$$

So the distance between any two vectors is simply the linear combination of the basis vectors.

Linear Distances Within a Hexagonal Space

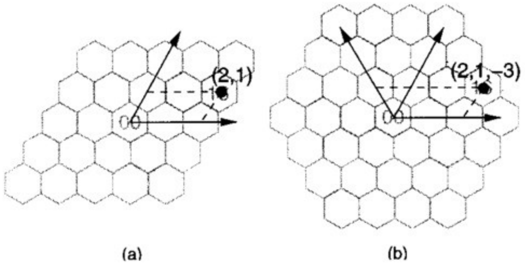
Any point within the hexagonal grid acts like a vector where the distance d between any two points is d = a + b. As a is x units from the y axis and b is y units from the x axis, the point (x, y) is on the z axis. This is easier than in a square grid where the distance d = $\sqrt{(x^2 + y^2)}$. Also, the distance from the origin out to a point in square grid is not always the same, in other words, not on the locus of a circle. It could be on the locus of a circle, but not always. On the other hand in a hexagonal grid, the distance from the origin out to another hexagon is always the same.



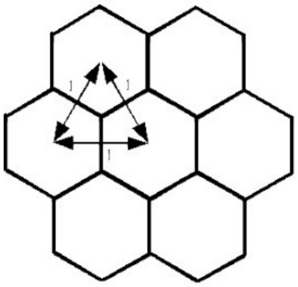


The hexagonal grid is more flexible, and you don't have to deal with square roots.

It should be the role of mathematicians to make math simple, and this does it. Each point is found on the outer side of the hexagon which is split into the coordinates of that point. The slope of the line from one point to another depends upon the angle of the line, and the length of the line, always z = x + y, and always at an angle θ, which can be expressed as z′ = x cos θ + y sin θ. That will give any slope. The coordinate/line segments are rotated.



(a)
(b)



A hexagon, surrounded by its equidistant neighbours.

*thanks to http://devmag.org.za/ for clear information concerning hexagonal coordinates

Hexagonal Grid Coordinates To Pixel Coordinates

This question talks about generating the coordinates themselves, and is quite useful. My issue now is in converting these coordinates to and from actual pixel coordinates. I am looking for a simple way to find the center of a hexagon with coordinates x, y, z. Assume (0,0) in pixel coordinates is at (0,0,0) in hex coordinates, and that each hexagon has an edge of length s. It seems to me like x, y, and z should each move my coordinate a certain distance along an axis, but they are interrelated in an odd way I can't quite wrap my head around it.

Bonus points if you can go the other direction and convert any (x, y) point in pixel coordinates to the hex that point belongs in.

For clarity, let the "hexagonal" coordinates be (r, g, b) where r, g, and b are the red, green, and blue coordinates, respectively. The coordinates (r, g, b) and (x, y) are related by the following:

$$y = 3/2 * s * b$$

$$b = 2/3 * y / s$$

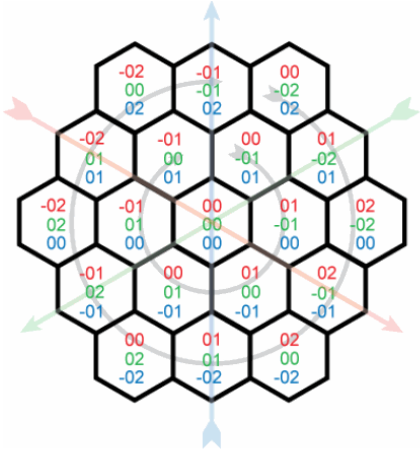
$$x = \surd(3) * s * (b/2 + r)$$

$$y = - \surd(3) * s * (b/2 + g)$$

$$r = (\surd(3)/3 * x - y/3) / s$$

$$g = -(\surd(3)/3 * x + y/3) / s$$

$$r + g + b = 0$$



Derivation

I first noticed that any horizontal row of hexagons (which should have a constant y-coordinate) had a constant b coordinate, so y depended only on b. Each hexagon can be broken into six equilateral triangles with sides of length s; the centers of the hexagons in one row are one and a half side-lengths above/below the centers in the next row (or, perhaps easier to see, the centers in one row are 3 side lengths above/below the centers two rows away), so for each change of 1 in b, y changes 3/2 * s, giving the first formula. Solving for b in terms of y gives the second formula.

The hexagons with a given r coordinate all have centers on a line perpendicular to the r axis at the point on the r axis that is 3/2 * s from the origin (similar to the above derivation of y in terms of b). The r axis has slope –√(3)/3, so a line perpendicular to it has slope √(3); the point on the r axis and on the line has coordinates (3√(3)/4 * s * r, –3/4 * s * r); so an equation in x and y for the line containing the centers of the hexagons with r-coordinate r is y + 3/4 * s * r = √(3) * (x – 3√(3)/4 * s * r). Substituting for y using the first formula and solving for x gives the second formula.

The set of hexagons with a given r coordinate is the horizontal reflection of the set of hexagons with a z coordinate, so whatever the formula is for the x coordinate in terms of r and b, the x coordinate for that formula with z in place of r will be the opposite. This gives the third formula.

The fourth and fifth formulas come from substituting the second formula for b and solving for r or z in terms of x and y.

The final formula came from observation, verified by linear algebra with the earlier formulas.
https://stackoverflow.com/questions/2459402/hexagonal-grid-coordinates-to-pixel-coordinates

$$u = x,\ v = \frac{1}{2}x + \frac{\sqrt{3}}{2}y, \text{ so } u^T v = [1, 0]\begin{bmatrix}\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{bmatrix} = \frac{1}{2}.$$

$$x = u + \frac{v}{2} \text{ and } y = \frac{\sqrt{3}v}{2}.$$

$$u = x - \frac{y}{\sqrt{3}} \text{ and } v = 2\frac{y}{\sqrt{3}}.$$

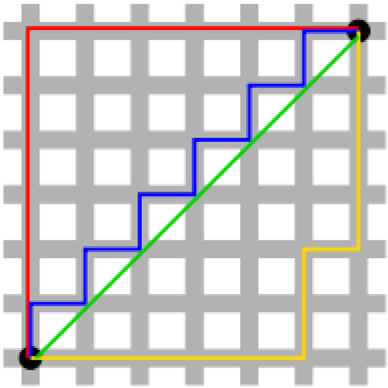
A coordinate system for hexagonal pixels

Wesley E. Snyder[1], Hairong Qi[1], William Sander[2]
[1]Center for Advanced Computing and Communication,
North Carolina State University, Raleigh, NC 27695-7914
[2] U.S. Army Research Office, Research Triangle Park, NC

The height of an equilateral triangle is the y side of a right triangle, i.e., h = s(√3)/2, and the x side of a right triangle is s/2; so for s = 1, s = $\sqrt{(((\sqrt{3})/2)^2 + (1/2)^2)}$ using the Pythagorean Theorem.

Taxicab Geometry

This is similar to taxicab geometry which depends upon the Cartesian 90º coordinate system. Between any two points on a straight line l is drawn two sides of an equilateral triangle such that there are stair-step triangles along l. One side of each triangle is a line segment of l. The other two sides of these triangles are called projections onto the coordinate system. The sum of these projections are added as vectors, one positive and one negative, so the distance of a line segment on l is the sum of the differences of these vector pairs.



As in taxicab geometry where circles are actually squares, a circle in hexagonal geometry is a hexagon. The geometric analog to π in taxicab geometry is 4, whereas in hexagonal geometry, it is 3. But the unit distance for both geometries is x + y = 1, where z = 1. For all values of z, z = x + y. (Absolute values are understood.)

The distance between any two points **v** and **u** with their origins at the center of the hexagon is the same as the distance of **v – u**:

$$d([x_1, y_1, z_1],[x_2, y_2, z_2]) = d([x_1 - x_2, y_1 - y_2, z_1 - z_2],[0, 0, 0])$$

$$= |x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|$$

The distance from the origin of the hexagon to any point within the hexagonal space can be calculated as half the sum of the absolute values of the coordinates. Thus,

$$d([x, y, z], [0, 0, 0]) = \frac{x + y + z}{2}$$

The Equations of a Line

There are three straight lines passing through the origin. These are called proper.

$$x = 0$$
$$y = 0$$
$$z = 0$$

Translating these lines so that each line intersect the same arbitrary point within the hexagonal plane,

$$x = k$$
$$y = k$$
$$z = k$$

where k is any combination of two of either x, y, or z and is a constant equal to the orthogonal distance from any of the axes X, Y, or Z respectively. Also,

all $x = k$ are parallel to Y,
all $y = k$ are parallel to X, and
all $z = k$ are neither parallel to X nor
Y, and thus are parallel to each other.

All lines $\xi = k$ are parallel to each other, where ξ, called the coordinates of the hexagonal plane, is either x, y, or z. If $\xi = 0$, then all $\xi = k$ are parallel. If any one of ξ is not parallel to the other two, then it is orthogonal to them. Orthogonality is due to a 60° angle between any two lines. If unit vectors **u**, **v**, and **w** are coincident to the lines $\xi = k$, whether they extend from the origin or any arbitrary point or from each one of the axes, they are linearly independent and form the basis for a vector space.

From $z = k$, we can write $x + y = k$, where k is the length of the side of a hexagon or the outer side of one of the six inner triangles within the hexagon.

Starting with two coordinates, P(a, b), we can find the third coordinate from

$$x + y + z = 0$$

where (a, b) is any combination of x, y, or z.

The Intersection of Lines

What if each coordinate is a different number such that

$$x = k$$
$$y = m$$
$$x = n \,?$$

Then because each of these lines cross and are orthogonal to the major axes X, Y, and Z, there is a point of intersection P(k, m, n).

Rotation

There is an easy trick to rotating a vector attached to the origin or a point rotating about the origin: simply rotate the coordinates from right to left and/or change the signs. For a 60° rotation:

$[x, y, z] \rightarrow [y, z, x]$, (rotate once)

$[y, z, x] \rightarrow [z, -x, y]$, (rotate twice and change 1 sign)

$[z, -x, y] \rightarrow [-x, -y, -z]$, (rotate twice and change 2 signs)

$[-x, -y, -z] \rightarrow [-y, -z, -x]$, (rotate twice) and

$[-y, -z, -x] \rightarrow [-z, x, -y]$. (rotate twice)

(Removing the z coordinate, we have $[\mathbf{x}, \mathbf{y}] \rightarrow [-\mathbf{y}, \mathbf{x} + \mathbf{y}]$.)

By applying the same type of rotation from left to right, we can obtain angles of 120, 180, 240, and 300 degrees. Generalizing, we can rotate a point n×60° around any point **c**:

$$R_{60n}(\mathbf{v} - \mathbf{c}) + \mathbf{c}$$

This describes a translation of the center of a hexagon into another hexagon.

Reflection

To reflect a point about $x = 0$, the y-axis,

flip x, keep z, recalculate to $y = -x - z$.

To reflect a point about $y = 0$, the x-axis,

flip y, keep z, recalculate to $x = -y - z$.

To reflect a point about $x + y = 0$, the z-axis,

flip all three x, y, and z.

With suitable translations we can reflect a point about any line.

For example, to reflect a point about the line x = 1, we translate the point by [− 1, 0], reflect it about the y-axis, and then translate it back again by [1, 0].

$$R_Y(\mathbf{v} - \mathbf{e}) + \mathbf{e}$$

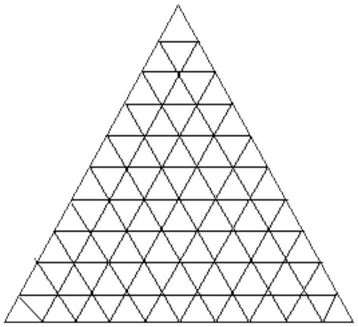
Chapter Three

Planar Measurement

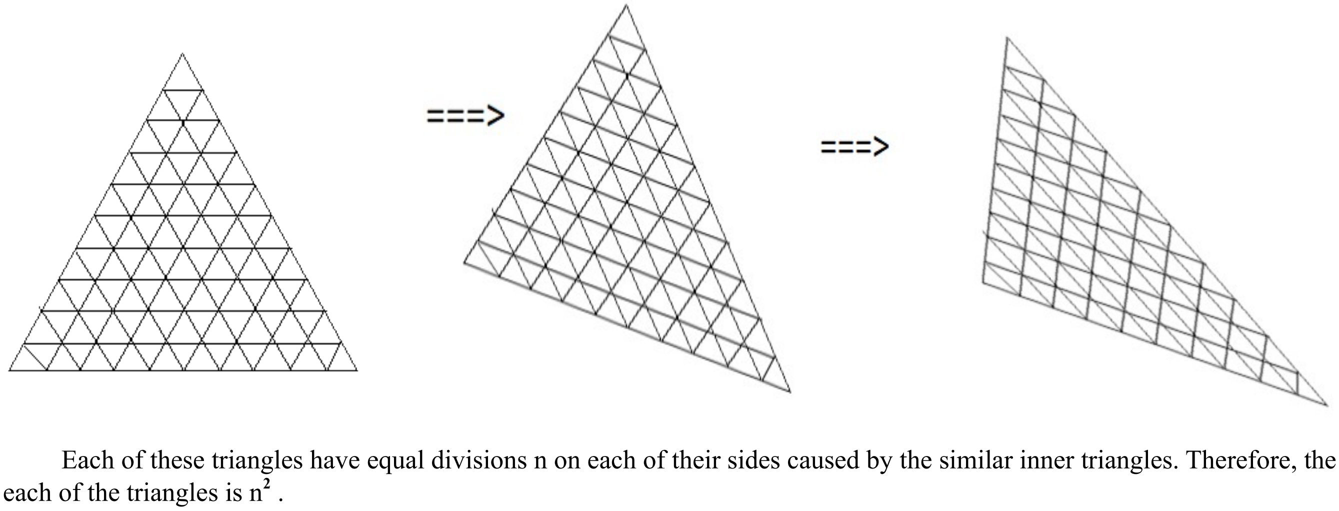
Triangling a Number

Triangles are more basic for measurement than are squares because the triangle is the most simple of the polygons. Any number n squared $n^2 = n \times n$. In Nature's Way of Measuring, it is called n triangled, because it is the multiplication of two sides of an equilateral triangle instead of the two sides of a square. Dividing all sides of a square by n and connecting each point to its opposite point with a line, the square is divided into n^2 (squared) squares. If the sides of an equilateral triangle are divided by n, and each point is connected to its two opposite points (at 60^o angles) with a line, the triangle is divided into n^2 (triangled) triangles. The number n has been triangled.

The Area of a Triangle



The square of a number n, that is, n^2, has a one–to–one correspondence with the triangle having an area of n^2. Dividing a square into n^2 similar squares, is the same number when you divide an equilateral triangle into n^2 similar triangles. In the figure, $10^2 = 100$ triangles. The triangle of a number is the area of an equilateral triangle. This can be generalized into any triangle. Also, if nm is the area of any rectangle, then ½ nm is the area of a triangle where n is the base and m is the height of the side opposite the hypotenuse of a right triangle. But we can substitute the area of any triangle with its equivalent area in an equilateral triangle, taking n as the divisor of any side, then n^2 is the number of similar triangles within that triangle. So ***the area of any triangle can be expressed as n^2 similar triangles***. Remember that. Though a triangle has sides of different lengths, an equilateral triangle of equal area can be stretched to create that triangle.



Each of these triangles have equal divisions n on each of their sides caused by the similar inner triangles. Therefore, the area for each of the triangles is n^2 .

To be more specific, any triangle can be divided into n^2 similar triangles. What I call a similar triangle is the smaller triangle with the same inside angles corresponding to the angles of the larger triangle. Therefore, the lengths of the sides of the larger triangle are whole number multiples of the sides of the smaller triangles in corresponding places. It is as though the triangles shrink, keeping the same corresponding inside angles, and the corresponding sides keep the same ratios to each other. So the left side of a triangle is divided by the sides of the smaller triangles; the base is divided by the bases of the smaller triangles, and the right side is divided by the right sides of the smaller triangles bordering that side. It is as though the above equilateral triangle was stretched in one or more directions.

Triangular Numbers

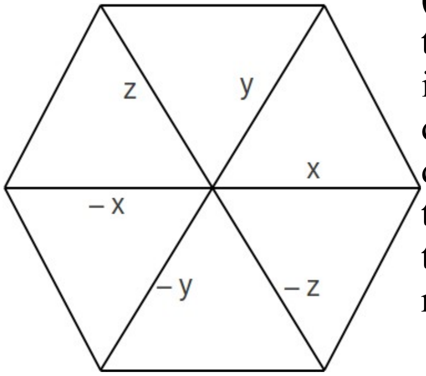
The number of triangles on each level of the above triangle is 1, 3, 5, 7, 9, …, $2n – 1$, where n is the level of the triangle, or the length of the side of the triangle. Another series where the intersecting points are added together, i.e., 1, 3, 6, 10, 15, …, $n(n + 1)/2$, where $n = 1, 2, 3, 4, ...$ is usually called triangular numbers. But adding the number of inner triangles or adding up each level of triangles, we get the series, 1, 4, 9, 16, …, n^2. I would define triangular numbers as how many unit triangles are within an equilateral triangle. This number is the same as a square because adding up two consecutive triangular numbers is a square, literally.

Finding the Triangular Root of a Number

Let an equilateral triangle be divided into n^2 similar triangles. By the definition of triangling a number n, each side of the triangle is divided into n parts. Therefore, the triangular root $\sqrt{(n^2)} = n$. Now if $t = n^2$, then $\sqrt{t} = n$. The triangular root of t is equal to n, where t is the number of similar triangles within an equilateral triangle with sides measuring n units. The triangular root becomes the scale of any triangle. In fact, the triangular root of any number is the length of a line, where as the triangle of a number is an area.

The triangular root of an area becomes a line

The triangular root of an area becomes a line. This is true whether the area is $(x + y)^2$ or $(xy)^2$. So if a binomial is an area, the triangular root of it is a line. If taking an ordinary second degree equation representing something in two dimensions, taking the triangular root of it changes it to one dimension. It would seem that a similar operation on three dimensions such as a cube would flatten the three dimensions into two-dimensional space such as a hexagon which has six equilateral triangles with axes x, y, and z. The three axes inside a hexagon represent the three spacial dimensions of the cube.

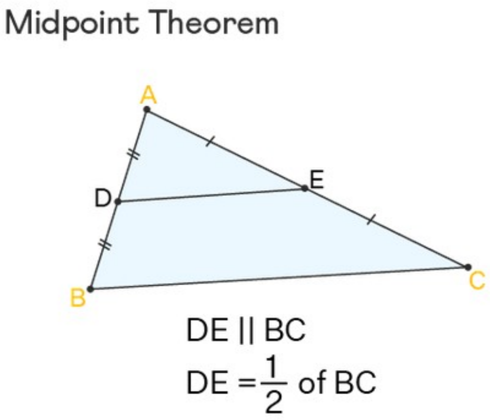


What is the Midpoint Theorem?

(from https://www.cuemath.com/geometry/mid-point-theorem/)

The midpoint theorem states that "the line segment joining the midpoints of any two sides of a triangle is parallel to the third side and equal to half of the length of the third side". It is often used in the proofs of congruence of triangles.

Consider an arbitrary triangle, ABC. Let D and E be the midpoints of AB and AC respectively. Suppose that you join D to E. The midpoint theorem says that DE will be parallel to BC and equal to exactly half of BC. Look at the image given below to understand the triangle midpoint theorem.



Midpoint Theorem Proof:

Compare ΔAED with ΔCEF:

Statement	Reason
1. AE = EC	E is the midpoint of AC (Given)
2. ∠DAE = ∠FCE	alternate interior angles
3. ∠DEA = ∠FEC	vertically opposite angles
4. ΔAED ≅ ΔCEF	By the Angle-Side-Angle criterion
5. DE = EF and AD = CF	By CPCTC
6. AD = BD	D is the midpoint of AB (Given)
7. BD = CF	From 5 and 6
8. BCFD is a parallelogram.	BD || CF (by construction) and BD = CF (from 7)
9. DF || BC and DF = BC	BCFD is a parallelogram
10. DE || BC	DE is part of DF and from 9
11. DE + EF = BC	E is a point on the line segment DF
12. 2DE = BC	DE = EF from 5
13. DE = 1/2 × BC	Dividing both sides by 2

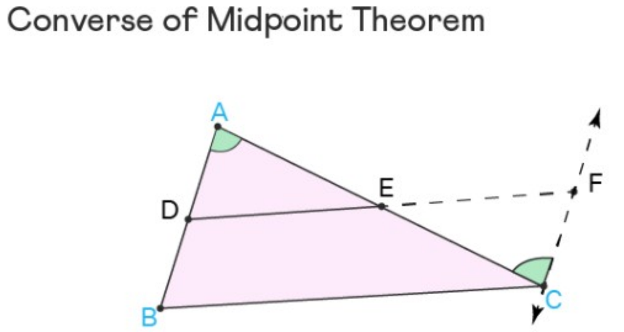
The midpoint theorem is proved by 10 and 13

Converse of Midpoint Theorem

Statement: The converse of the midpoint theorem states that "the line drawn through the midpoint of one side of a triangle that is parallel to another side will bisect the third side". We prove the converse of mid point theorem by contradiction.

Proof of Mid Point Theorem Converse

Consider a triangle ABC, and let D be the midpoint of AB. A line through D parallel to BC meets AC at E, as shown below.

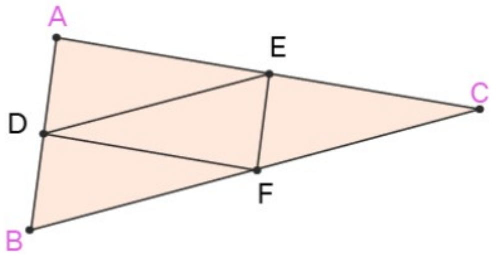


Proof of Converse of Midpoint Theorem

Statement	Reason
1. BCFD is a parallelogram	DE || BC (given) and BD || CF (by construction)
2. BD = CF	Opposite sides of a parallelogram are equal
3. AD = BD	D is the midpoint of AB (given)
4. AD = CF from 2 and 3	from 2 and 3
Compare ΔAED with ΔCEF:	
5. ∠DAE = ∠ECF	Alternative angles
6. ∠DEA = ∠FEC	Vertically opposite angles
7. ΔAED ≅ ΔCEF	By AAS criterion (using 4, 5, and 6)
8. AE = CE	By CPCTC

Application of Midpoint Theorem, or finding the area of a triangle

A consequence of the midpoint theorem is that if we join the midpoints of the three sides of any triangle, we will get four (smaller) congruent triangles, as shown in the figure below:



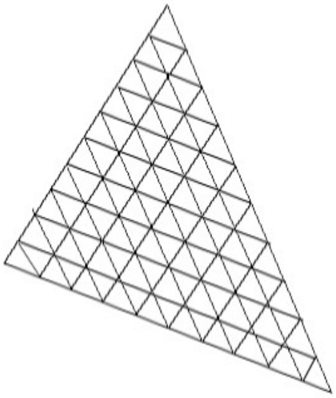
We have: ΔADE ≅ ΔFED ≅ ΔBDF ≅ ΔEFC.
Proof: Consider the quadrilateral DEFB. By the midpoint theorem, we have:

DE = 1/2 × BC = BF
DE || BF

Thus, DEFB is a parallelogram, which means that ΔFED ≅ ΔBDF. Similarly, we can show that AEFD and DECF are parallelograms, and hence all four triangles so formed are congruent to each other.

Generalizing the Application of the Midpoint Theorem

As any triangle can be given the properties of a regular equilateral triangle by subdividing each side by a number n, called the meta-length, and connecting the division points on one side to the corresponding division points on the other two sides, getting something like this,



The count of the number of inner triangles is always n^2. Therefore taking the length n of a chosen side, the area A of the triangle will be $A= n^2$. (and that is pronounced n triangled) Measurement is made in pure number, and the unit of measurement has to be decided beforehand.

The Binomial as Coordinate

Changing the signs within the binomial will give you the different sextants of the hexagon. Based upon z = x + y, you can access the

(x, y, z) sextant in the $(x + y)^2$ binomial, the
(y, z, x) sextant in the $(y + z)^2$ binomial, the
(z, – x, y) sextant in the $(z – x)^2$ binomial, the
(– x, – y, – z) sextant in the $(– x – y)^2$ binomial, the
(– y, – z, – x) sextant in the $(– y – z)^2$ binomial, and the
(– z, x, – y) sextant in the $(x – z)^2$ binomial.

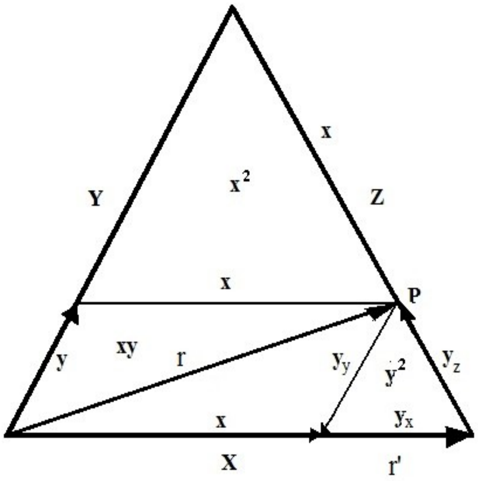
This can be done by taking the triangular root of a binomial. It will give you one of the axes of the hexagon which represents a flattened 90° coordinate system.

Therefore,

$$z = \sqrt{}[(x + y)^2]$$
$$x = \sqrt{}[(y + z)^2]$$
$$y = \sqrt{}[(z - x)^2]$$
$$-z = \sqrt{}[(-x - y)^2]$$
$$-x = \sqrt{}[(-y - z)^2]$$
$$-y = \sqrt{}[(x - z)^2]$$

Just as taking the triangle of one of the axes of a hexagon gives you a binomial, such as $c^2 = (a + b)^2$, taking the triangular root of a binomial is the equivalent of flattening two dimensions into one dimension.

Using the Binomial to Find the Length of a Vector



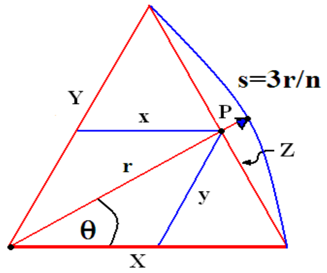
I had an epiphany. Is there a parallel to the Pythagorean Theorem in the 60° Coordinate System? Drawing an angled line within the equilateral triangle, I wanted to know its length. I noticed the different triangles and their relationships with the parallelogram enclosed in the triangle.

I have already come up with the idea that any line segment c = a + b, where x and y are the coordinates of any point on c. A similar idea is that a vector **r** has components **x** and **y** such that **r** = **x** + **y**. Measuring the length z_r of **r**, I came up with $z_r = 3+19/32$ or 3.59375 inches. Next, I defined this vector as the diagonal of a parallelogram with sides x = 3 inches and y = 1 inch. Using the Pythagorean Theorem, $\sqrt{}(3^2 + 1^2) = \sqrt{}(10) = 3.16228$ inches. That doesn't work. But what I do know is that

$$z_r = x + y, \text{ and}$$
$$z_r^2 = (x + y)^2, \text{ but}$$
$$(x + y)^2 = x^2 + xy + y^2.$$

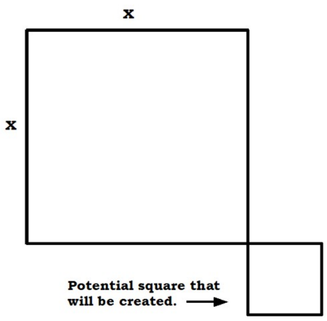
One of the main ideas of the 60° Coordinate System is that a side of an equilateral triangle triangled is the area of the triangle. So, if I know the area of the equilateral triangle whose side is the line z_r I drew, I can take the triangular root of the area and get the length of the line. Since z_r is the length of the line, $z_r^{\ 2}$ is the area of the equilateral triangle whose side is z_r, and by the above equations, $z_r^2 = x^2 + xy + y^2$. Therefore, the length of the line is $z_r = \sqrt{}(x^2 + xy + y^2)$. I plugged in the values of x = 3 and y = 1 and came up with $z_r = \sqrt{}(3^2 + 3 + 1^2) = \sqrt{}(13) = 3\ 19/32$ inches. Voila! It works according to my ruler measurements.

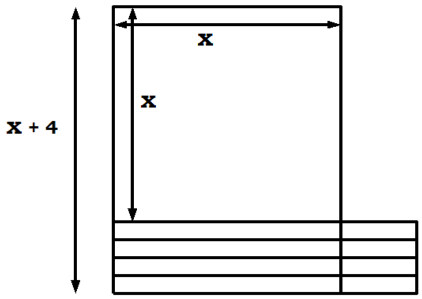
You can do this, but remember that $z_r = x + y$ is simply $z = x + y$ and if the vectors **x** and **y** are added, then **r** = **x** + **y** where **r** is equal in length to a side of an equilateral triangle, no matter the angle of **r**. Although x and y are the coordinates of a point P(x, y) where **r** crosses Z, **x** and **y** are the vector components of **r**, such that **x** = **r** cos θ, and **y** = **r** sin θ, where 0 < θ < 60° and x and y are the lengths of the sides of the triangles whose areas are x^2 and y^2, both encased in YXZ.



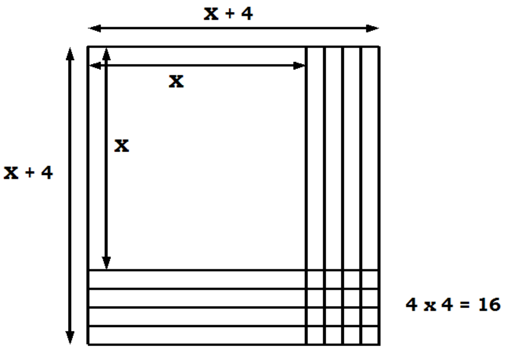
Quadratic Equations = Areas, Completing the Triangle

Completing the square is a way to solve quadratic equations. This was done anciently by the Greeks by a process of increasing the area of a square by adding unit width rectangles to two sides of the square in such a way as the overlapping rectangles created a smaller square connected to the corner of the original square.





Each rectangle is one unit in width and X + 4 units in length.



When adding 4 more rectangles the resulting square has 16 small squares inside it.

For example, the resulting quadratic equation $y = (x + 4)^2$ comes from $y = x^2 + 8x + 16$. Starting with the original square, x^2, we add two (4x)'s. Then 2(4x) = 8x, or two rectangles. Now the equation can be solved for x. But this equation was obtained by completing the square of another equation, $x^2 + 8x - y = 0$. Adding y to both sides and adding the square of ½ 8 to both sides, that is, $4^2 = 16$.

$x^2 + 8x + 16 = y + 16$, and solving for x,

$$(x + 4)^2 = y + 16,$$

$$x + 4 = +/- \sqrt{(y + 16)}$$

$$x = - 4 +/- \sqrt{(y + 16)}$$

Proof: $x^2 + 8x + 16 = (x + 4)^2$

Generalizing this,

$$x^2 + bx - y = 0$$

$$x^2 + bx + b^2/4 = y + b^2/4$$

$$(x + b/2)^2 = y + b^2/4$$

$$x + b/2 = +/- \sqrt{(y + b^2/4)}$$

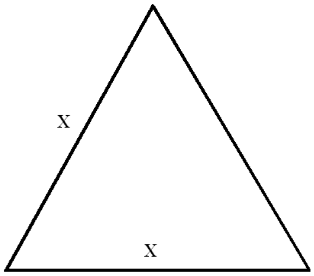
$$x = - b/2 +/- \sqrt{(y + b^2/4)}$$

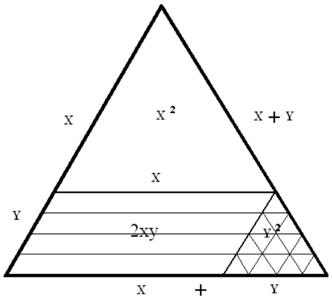
Using the equilateral triangle in a similar way, starting with a triangle having sides of unknown length x, the length of the triangle is extend y units so that each side of the triangle is x + y units. The corner of the larger triangle whose side's length is x + y, and whose area is $(x + y)^2$, consists of a smaller triangle that has sides y units in length with an area of y^2. A parallelogram with sides whose lengths are x and y with an area 2xy filling the rest of this triangle. The resulting total area of the triangle is $x^2 + 2xy + y^2$.

Completing the Triangle

Completing the triangle is similar to completing the square, but using triangles instead of squares. The number of triangles you start with is x^2. What you add is n rows of 2x unit triangles (because half of the triangles are pointing in the opposite direction) plus n^2 unit triangles. So the area becomes $x^2 + 2xn + n^2$. If we allow y = n, the area of the completed triangle is equal to the binomial $(x + y)^2$. Thus we can write

$$(x + y)^2 = x^2 + 2xy + y^2.$$





This is the same as using squares because there is a one-to-one correspondence between a square and an equilateral triangle. The geometry reflects the difference in adding increased area.

There is one anomaly. When an equilateral triangle is divided into two smaller equilateral triangles x and y and a parallelogram filling in the space between them. The area of the parallelogram is 2xy equilateral triangles. While computing the area of the equilateral triangle, adding up all the smaller unit equilateral triangles you get $x^2 + 2xy + y^2 = (x + y)^2$. Yet, when computing the length of a line r from one corner of the parallelogram to its opposite corner, $r = \sqrt{(x^2 + xy + y^2)}$. It looks very similar to the binomial. It seems that you use 2xy when calculating *areas* and xy when calculating *lengths*. The difference is that one is a binomial and the other is a length. We must not get the two confused. Yet, the length triangled is a volume. Surely, $\sqrt{(x^2 + 2xy + y^2)}$ and $\sqrt{(x^2 + xy + y^2)}$ are two different lengths. The first seems to be the side of an equilateral triangle, and the other, the diagonal of the inner parallelogram.

A Really Abstract Generalization of the Binomial

Let ξ represent the addition of values such as x + y, or the multiple xy of a value. If ξ represents xy, then ξ can expand laterally into a length $x^2 + \xi + y^2$ or if ξ represents x + y, ξ can expand both laterally and vertically into ξ^2, and into a volume, $x^2 + 2\xi + y^2$.

Therefore, $\xi \rightarrow x^2 + xy + y^2$ and $\uparrow\xi, \rightarrow \xi^2 \rightarrow x^2 + 2xy + y^2$.
Theorem: If ξ represents a line, then ξ^2 represents a plane.
Any line triangled is a plane.

The Relationship Between Any Triangle and an Equilateral Triangle

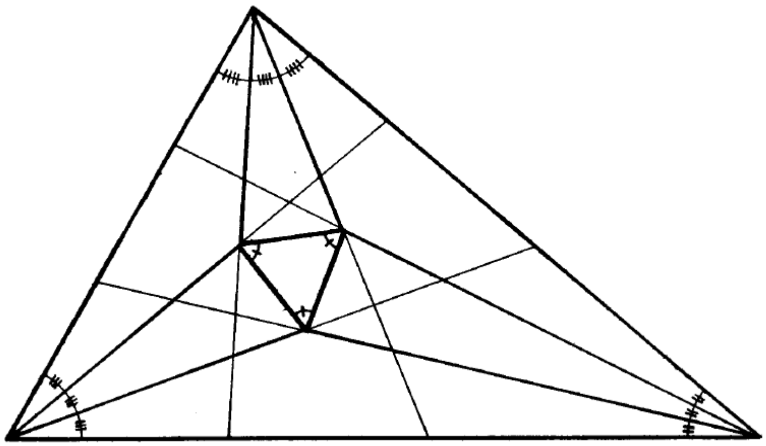


Fig. *100.51 Morley Theorem:* The trisectors of any triangle's three angles describe an equiangular triangle.

Morley's Theorem says that the trisectors of the angles of any triangle describe an equilateral triangle. Therefore there is a relationship to any triangle and the equilateral triangle. There is a definite relationship to the divisions of angle in any triangle and to the face of the equilateral triangle within the given triangle. They have the same divisions. The divisions of angle the equilateral triangle is facing is reflected on that side of that equilateral triangle.

Likewise, any triangle with a given area can be represented by an equilateral triangle having the same area. Dividing one side of an equilateral triangle into x equal segments, x^2 is equal to the area s of the triangle and the triangular root $\sqrt{s}$ of the area s is equal to the length x of one side. Any triangle with an area s having each side divided into x segments can be represented by a corresponding equilateral triangle of the same area s. If any triangle has all sides divided into x segments each, and lines are drawn connecting opposite segments, then that triangle is divided int x^2 triangles. The corresponding equilateral triangle whose sides are divided into x segments is also divided into x^2 triangles. Thus there is a one-to-one correspondence between the equilateral triangle and any given triangle.

The Relationship Between the Area of a Triangle and its Perimeter

The perimeter p of any triangle is $3\sqrt{s}$. This relationship between the area of the triangle and its perimeter can be extended to all polygons regular and irregular. Each polygon can be divided into triangles. Regular polygons, into similar triangles. So from a single triangle with a perimeter of $p = 3\sqrt{s}$, going outward from the center of any polygon to the perimeter,

for a square, $p = 4\sqrt{s}$,
for a pentagon, $p = 5\sqrt{s}$,
for a hexagon, $p = 6\sqrt{s}$,
and so on for any regular polygon of n sides, $p = n\sqrt{s}$.

Since $x = \sqrt{s}$, the perimeter p of each polygon is equal to nx.

Chapter Four

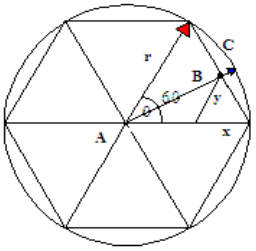
Circular Measurement

Getting Rid of π

A circle traditionally represents a cycle. The measure of a cycle is in π units which is the ratio of the radius to circumference of the circle. Just as 2π represents one cycle, so the sides of a hexagon having six unit sides, represents one cycle in a hexagonal coordinate system. So 6 units represents one cycle. Half of the perimeter of the hexagon is the unit of cyclic measure instead of π. In other words, the numeral 3 is the replacement of π. So n/3 is the new circular measurement, based, not on the circumference of a circle, but on the 6 chords of a circle as it encases a hexagon.

Consider the arcs of a circle. Let the hexagon be projected onto an inscribing circle such that one chord of the hexagon is equal to one arc of the circle. Then there are the six arcs cut from the circumference of a circle. These can be used for measuring degrees, whereas the chords can be used for radian measure.

The Natural Way of Measuring a Circle



If the hexagonal coordinate system is circumscribed where each corner of the hexagon touches the perimeter of a circle, we can get rid of π. The hexagon provides 6 chords that divides the circle into 6 arcs. Instead of trying to lay the radius of the circle out across its circumference where it covers an incomplete number of times, it is more logical to divide the circle into 6 arcs to describe a cycle or a part of a cycle. Therefore, the trigonometric functions do not depend upon π, but upon a rational number. An example has been given of using one geometric shape to calculate the volume of another. It may be possible to do that with other shapes such as the circle and sphere. The secret is to use the unit triangle or the unit tetrahedron.

The traditional way of measuring the area of a circle is to use π. For a unit circle where r = 1, the conventional area πr^2 is π = 3.14159 squares. The area of a unit equilateral triangle (each side is equal to one) is ½ hb = ½ (√3)/2 = (√3)/4 = .433013 squares/triangle. To change the area in squares to triangular units, take the reciprocal of .433013 squares/triangle. It changes to 2.309401 triangles/square. Multiplying the unit circle of 3.14159 squares by 2.309401 triangles/square, we get 7.255197 or 7 ¼ equilateral triangles. (Multiplying that by the synergetics constant for two dimensions, 9/8, we get a rational number of 8 equilateral triangles, so something must be wrong with its use here.) The space between the 6 chords of the hexagon encased by the circle and the 6 arcs of the circle over the hexagon is 7 ¼ – 6 = 1 ¼ equilateral triangles. 1¼ divided by 6 is 1/6 + 1/24 triangle or 5/24ths of a triangle in each area between the arcs and chords.

Area = 7 ¼ r^2 equilateral triangles for any circle. There is no **π** involved!

Let the circumference of a unit circle be defined as 6 arc lengths based upon the chords of the inscribed hexagon instead of the traditional 2πr. One sixth of the circumference is equal to 2πr/6. When r = 1, 2πr/6 = 1.047197. Multiplying that by the conversion factor of 1.06066, we get an arc length of 1.110720 or just 1. So letting π = 3, and, one arc will be 1r so the circumference will be 6r arcs.

It has been discovered that each circle has an area 7 ¼ equilateral triangles no matter what the radius is equal to. The equilateral triangle is scaleless. A circle's area of the next higher integral radius or frequency is just r^2 x 7 ¼. Here are some examples.

Circle of Radius r or Frequency	**Area in Squares of One Equilateral Triangle in the Hexagon**	[&]**Number of Unit Equilateral Triangles in One Sixth of the Hexagon**	**Number of Equilateral Triangles in a Circle Equals**	**Area in Unit Equilateral Triangles**
r = 1	$\sqrt{3}$ /4	$1^2 = 1$ x	7 ¼ =	$7^{1}/_{4}$
r = 2	$\sqrt{3}$ /2	$2^2 = 4$ x	7 ¼ =	29*
r = 3	3 $\sqrt{3}$ /4	$3^2 = 9$ x	7 ¼ =	$65^{1}/_{4}$
r = 4	$\sqrt{3}$	$4^2 = 16$ x	7 ¼ =	116#

* surface area of unit sphere (4 great unit circles are used to find the surface of the sphere)

\# surface area of sphere with r = 2

[&] Each major triangle in a hexagon is split into r^2 unit triangles. These are the triangular numbers, adding each layer one at a time to form the area of the triangle in equilateral triangles.

The triangular area of a circle divided by its radius triangled r^2 gives you the number of equilateral triangles within the circle, which is always 7 ¼. That is, there are always r^2 of them. So the new equation of the area of a circle without using π is A_{circle} = 7 ¼ r^2 triangles.

Let's Do This Again

Starting out with π as the area of the unit circle, multiply it by the synergetics constant for two dimensions. That is 3.14159 x $^{9}/_{8}$ = 3 ½ squares. Multiply this by the conversion factor of 2.309401 triangles/square and you get 8 triangles. It looks enticing, but if you subtract the 6 triangles of the inscribed hexagon, you get 2 which is supposed to be the area between the cords and the arcs. Multiplying 3 ½ by 2 (because there are 6 triangles in a hexagon) we get 7. So is it 7, 7 ¼ , or 8 triangles in a circle? Is the space between the arcs and the chords 1, 1 ¼ , or 2?

The area of one unit equilateral triangle is .433013 squares. Six of those is 2.598078 squares, which is the area of a unit hexagon. Subtract that from a unit circle, which is π, and you get .543515/square. That is the area between the cords and the arcs in squares. So to change that to triangular measure by multiplying that by 2.309401 triangles/square and you get 1.25 or

1 ¼ equilateral triangles. Visually, you can see only 6 unit equilateral triangles within a unit circle. But what is the area between the arcs and the chords, and how much of that fits into an equilateral triangle? That is the only thing that counts here. By my calculations it is 1 ¼ equilateral triangles. Now 6 equilateral triangles in a hexagon + 1 ¼ = 7 ¼. There are 7 ¼ equilateral triangles in the unit circle.

It is notable that the radius triangled is the number of unit equilateral triangles or unit areas inside one sixth of the hexagon inscribed by the unit circle because that is the definition of triangling a number or the triangular root of a number being the area.

Using Nature's Way of Measuring, the unit of measure for area is only one unit equilateral triangle.

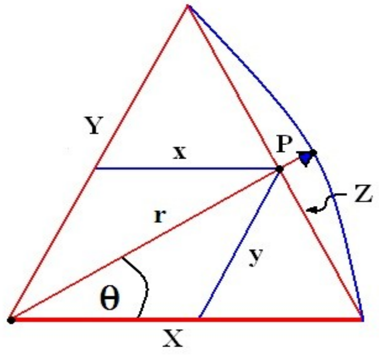
Surface Area of a Sphere

Since the surface area of a unit sphere is 4 great circles times the area of one of the great circles, 4 x 7 ¼ = 29 is the surface area of a unit sphere in equilateral triangles. Then the surface area of any sphere is 29 r^2.

Circular Measurements

The circumference of a circle $C = 2\pi r$.
Arc length $s = r\,\theta$.
To change from degrees to radians, $\theta = \pi/n$, or divisions of π.
Therefore, in radians, arc length $s = r\,\pi/n$.

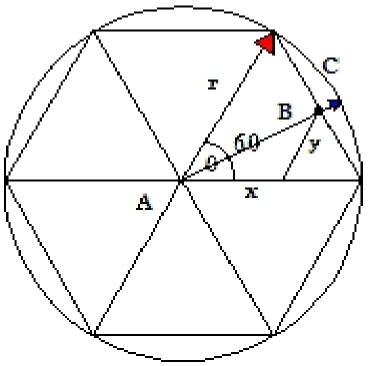
Now if we base π on the perimeter of the unit hexagon instead of on the circumference of the unit circle, and since the circumference of a circle is divided by a hexagon into six arcs, let $\pi = 3$. Where π would be, there are 3 chords.



Substituting 3 for π, arc length $s = 3\,r/n$.

Let $\pi = C/2r$. The circumference of the circle now becomes $C = 2 \cdot 3r = 6r$. It isn't how many times the radius fits around the circumference, but 6 chords of the circumference times the radius outwards, the circumference expanding.

A Review of Trigonometry



The figure to the left shows a hexagon. It represents the cycle of the circle based upon the chords of the hexagon instead of the ratio π of the circumference to the diameter. Using a unit triangle where each side is one unit, and therefore, each chord is one unit, the circumference of the circle is divided into 6 parts. Let one rotation of **r**, beginning at the x-axis, around the circumference and back to the x-axis stand for one cycle. Each chord represents 1/6th of a cycle. Then it is logical to keep that division of 6 and divide each chord into 6 equal parts or 1/36th of the hexagon. Then there would be 36 divisions of the circle. If there are then 10 divisions between each 1/36th mark, there will be 360 divisions around the hexagon. Extending those divisions to the enclosing circle, the circle then receives 360 divisions. The angle θ can be represented by these divisions, being projected onto the circle, which would be 360°. Divisions on the hexagon would be 360 radians.

Note: The ancient Sumerians of the Mesopotamian area knew of the procession of the equinoxes to be 72 years for one degree of procession. To them, this knowledge was simply a gift from the gods. Then using the sacred hand, a span of 5 fingers, 72° x 5 = 360°. So to them 360° is a span of the heavens, a span meaning a complete cycle. We have kept this from them.

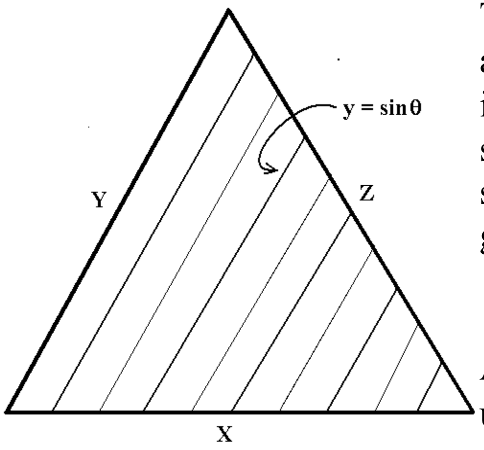
Using radians, a trigonometric table within the unit triangle of the hexagon using radian measure based upon the divisions of a circle would be n/6, n/36, or, n/360.

The definition of trigonometric functions are based upon $z = x + y$: $\sin s = y$, $\cos s = x$, and $\sin s + \cos s = 1$. We will allow $\tan s = z$.

Whereas in a 90° coordinate system the hypotenuse of a right triangle or the radius of a unit circle determines cos θ and sin θ as it rotates from 0° to 90°, and the radius $r = \sqrt{(x^2 + y^2)}$, the X side of an equilateral triangle or radius of a unit circle determines the cos θ and sin θ as it rotates from 0° to 60°, and the radius $r = x + y$. Also, $\cos \theta + \sin \theta = 1$. In the 90° coordinate system $\cos^2 \theta + \sin^2 \theta = 1$, and so we continually see that the 60° coordinate system simplifies these trigonometric functions as well as many other functions.

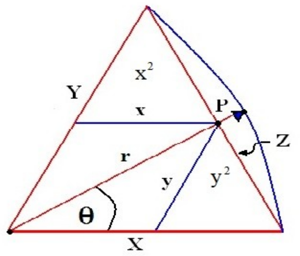
The divisors of 360 include all the digits except 7, but only 6 and 60 divide 360 in a symmetry that the other digits don't. This is because $36 = 6^2$. 360/2 = 180, 360/20 = 18; 360/3 = 120, 360/30 = 12; 360/4 = 90, 360/40 = 9; 360/5 = 72, 360/50 = 7.2; 360/6 = 60, 360/60 = 6; 360/8 = 45, 360/80 = 4.5; 360/9 = 40, 360/90 = 4.

The only divisor here that creates symmetry is 6 because 360/6 = 60 and 360/60 = 6 where the 6 and the 60 are interchangeable and there are no other digits you can do this with. Therefore, it seems more natural to divide the circumference of a circle into 6 sections, or multiples of 6.



There are 12 * 30° segments in a circle and 5 * 72° segments in a circle. Every 30° is divided into 5 * 6°. If the circle is divided into 12° segments, every 5th segment is 60°. Half of each 12° segment would be 6°, so dividing the circle into 6 degree segments, you get the numbers 5, 6, 12, 36, 60, 72, and 360, getting multiples of 2, 3, and 5.

Next, look at the trigonometric functions using degrees. A trigonometric table is based on a single unit triangle within a unit hexagon using degree measure θ. Dividing z into x and y, since $z = x + y$, each coordinate pair corresponds to a degree of arc located on a sixth of a circle. What is listed below is every five degrees from zero degrees to sixty degrees, where every division of y or x is divided by 60.



It must be remembered this is a description of the smaller triangles within the larger triangle, such that $\cos\theta + \sin\theta = 1$, or

$$\cos\theta = 1 - \sin\theta, \text{ or } x/(x + y)$$
$$\sin\theta = 1 - \cos\theta, \text{ or } y/(x + y) \text{ and}$$
$$\tan\theta = \sin\theta/\cos\theta.$$

When one function approaches zero, the other is approaching one. As the x triangle decreases, the y triangle increases and visa versa.

Trigonometric Table (based upon the division of a 60° arc into 60 segments, 1° = 1/60):

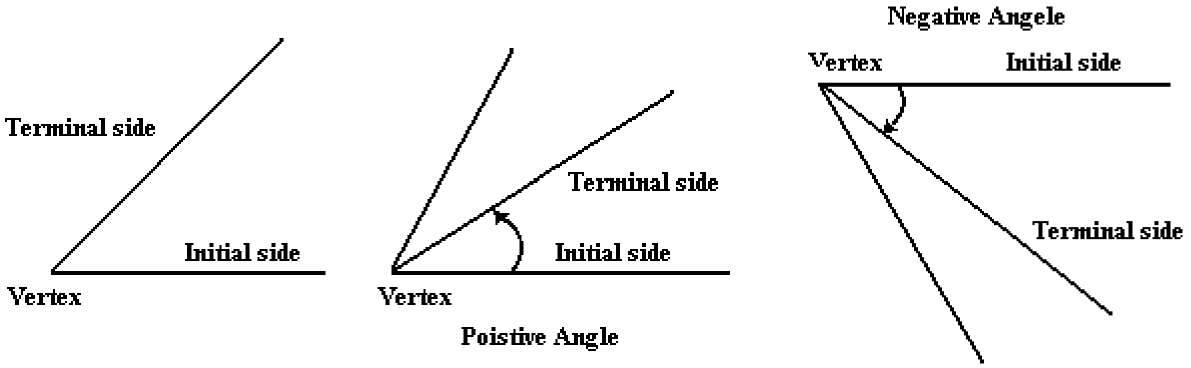
θ	***Sin θ***	***Cos θ***	***Tan θ***	***Cot θ***
0°	0.000000	1.000000	0	∞
5°	0.083(333...)	0.916(666...)	0.090909	11.000036
10°	0.166(666...)	0.833(333...)	0.199999	5.000018
15°	0.250000	0.750000	0.333(333...)	3.000000
20°	0.333(333...)	0.6666(666...)	0.500000	2.000000
25°	0.416(666...)	0.5833(333...)	0.714285	1.400001
30°	0.500000	0.500000	1	1
35°	0.583(333...)	0.416(666...)	1.400001	0.714285
40°	0.666(666...)	0.3333(333...)	2.000000	0.500000
45°	0.750000	0.250000	3.000000	0.333(333...)
50°	0.833(333...)	0.1666(666...)	5.000018	0.199999

55°	0.916(666...)	0.0833(333...)	11.000036	0.090909
60°	1.000000	0.000000	∞	0

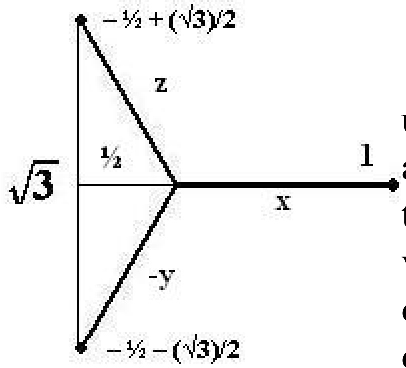
Sec θ = Sin θ or Cos θ because a secant is the chord underneath the arc. In other words,
Sec θ = x + y which is the z coordinate.

These values in this trigonometric table can be used for any problem within the hexagonal coordinate system.

Angles



Starting with the angle, it has an initial side, a terminal side and a vertex or the point of the angle. The standard position of the angle, called positive, is a counterclockwise rotation. The negative angle then has a clockwise rotation. An angle may be generated by making more than one revolution, the terminal side passing the initial side once or more than once. And depending upon the direction of rotation, a negative angle remains a negative angle, and a positive angle remains a positive angle.



Traditionally, trigonometric functions have been based upon the right triangle. The tetrahedral function $z^3 = +/- 1$ forms an interface between 90° and 60° coordinate systems. Based upon the right triangle, Sin 60° has been defined as (√3)/2, which is the volume of a unit equilateral triangle, and cos 60° as ½. The distance between the imaginary roots of $z^3 = 1$ is √3 which is also defined as tan 60°. The diagonal of a square with sides of √2 is sec 60° which is 2, the reciprocal of cos 60° . So, we have

sin 60° = (√3)/2,

cos 60° = ½,

tan 60° = √3 and

sec 60° = 2.

In a unit hexagon, any axis is the radius of the inscribing unit circle, and the hexagon is divided into 6 equilateral triangles. Choosing the upper right triangle, the x axis is the radius of the circle. As it swings upward from from 0° to 60° it cuts the z axis into two line segments. Call the upper line segment a whose length is $\cos \theta$, and the lower one b whose length is $\sin \theta$. It is so that

$$\cos \theta + \sin \theta = 1.$$

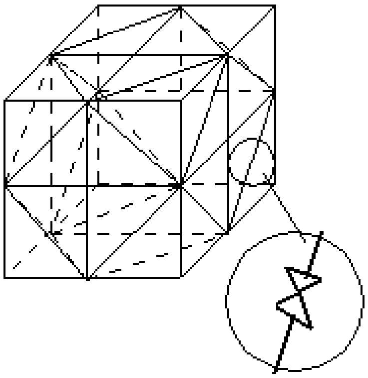
That means that as $\cos \theta$ increases in value from 1 to 0, $\sin \theta$ decreases from 0 to 1. All values, therefore, are from 0 to 1 as $0^\circ < \theta < 60^\circ$. So in a 60° coordinate system, we have

$\sin 60^\circ = 1$

$\cos 60^\circ = 0$

$\tan 60^\circ = \infty$

$\csc 60^\circ = 1$



Each diagonal is actually two vectors pointing to each other.

Take a cube where each side is divided into four squares. Let the edge of each of the 4 squares be $(\sqrt{2})/2$, making a cube having each edge be $\sqrt{2}$. Drawing one inch diagonals through each of the four squares on each face of the cube so that they form another square which is turned 45° from the square face, each of these diagonals can be connected to form 4 hexagons interlaced within the cube. Each of the diagonals is $\csc 60^\circ = x = y = z = 1$, and this shows the 45° angle between the 90° and 60° coordinate systems.

Imaginary numbers also form an interface between 90° and 60° coordinate systems. Using the side s of an equilateral triangle, an imaginary number is of type, $s/2 + h$, where s is the length of one side and h is the height of the triangle. For a unit equilateral triangle, the imaginary number would be $½ + (\sqrt{3})/2$. Since the height of the triangle would have as many divisions n as the side s, a general imaginary number would be $s/2n + h/n$. But since the height of a unit equilateral triangle is $(\sqrt{3})/2$, the imaginary number would be $s/2n + (\sqrt{3})/2n$. The distance between any two of $- s/2n - (\sqrt{3})/2n$, $+ s/2n - (\sqrt{3})/2n$, $- s/2n +(\sqrt{3})/2n$, $+ s/2n + (\sqrt{3})/2n$ would be $|(\sqrt{3})/n|$. In a 90° coordinate system, $\sin 60^\circ = (\sqrt{3})/2n$, and $\cos 60^\circ = s/2n$. The imaginary number is actually $+/- \cos 60^\circ +/- \sin 60^\circ$. That would be true whether it is in the 90° coordinate system or the 60° coordinate system. Therefore this imaginary number is an interface.

To turn $+/- (\cos 60^\circ = s/2n) +/- (\sin 60^\circ = (\sqrt{3})/2n)$ from a 90° coordinate system to a 60° coordinate system, n would have to be equal to $2/(\sqrt{3})$, and $s = 0$. The result would be $\sin 60^\circ =$

1 and cos 60° = 0. On the other hand, the sin 60° of the 60° coordinate system is equal to twice the cos 60° of the 90° coordinate system. This is due to the fact that, taking half of a unit equilateral triangle, which is a right triangle, with the angle opposite the height of the triangle equal to 60°, the base is one half the hypotenuse. The hypotenuse is the sine of the angle in the 60° coordinate system, whereas the base is the cosine of the angel in the 60° coordinate system. The height is not taken into consideration.

The sec of 60° is 2 which is the length of two connected sides of the unit hexagon. The tan of 60° is √3 which is the length of a line connecting the two ends of the two connected sides of the unit hexagon. With this information, a table of the trigonometric functions can be produced.

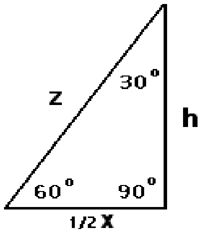
A trigonometric table can be based on a single triangle within a unit hexagon using degree measure θ. Even though degree measure is used, the result is the same as radian measure because it is based upon chords of the circle and not the circle itself.

θ	*Sin θ*	*Cos θ*	*Tan θ*	*Cot θ*
0°	0	1	0	∞
10°	1/6	5/6	(√3)/12, 1/5	5
15°	1/4	3/4	(√3)/8, 1/3	3
20°	1/3	2/3	(√3)/6, ½	2
30°	½	(√3)/2, ½	(√3)/4, 1	1
40°	2/3	1/3	(√3)/3, 2	1/2
45°	1/(√2), 3/4	1/(√2), 1/4	1, 3	1/3
50°	5/6	1/6	5(√3)/12, 5	1/5
60°	(√3)/2, 1	½, 0	(√3), ∞	0

Sec θ = Sin θ or Cos θ because a secant is the chord underneath the arc. In other words, Sec θ = x + y. (On the 30°, 45°, and 60°, as well as the Tan θ column, I have included measures from the 90° coordinate system.)

Also included in trigonometric measurement are the functions of secant, cosecant, and cotangent which are the reciprocals of cosine, sine, and tangent, respectively.

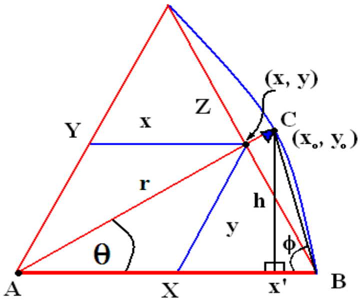
I include here a table of imaginary numbers obtained from the right triangle.



Z	*h*	*½ x*	*X*	*I-numbers*
1	(√3)/2	1/2	1	+/– ½ +/– (√3)/2
2	√3	1	2	+/– 1 +/– √3
3	3(√3)/2	1 1/2	3	+/– 1½ +/– 3(√3)/2
4	2(√3)	2	4	+/– 2 +/– 2√3
5	5(√3)/2	2 1/2	5	+/– 2½ +/– 5(√3)/2
6	3(√3)	3	6	+/– 3 +/– 3√3
7	7(√3)/2	3 1/2	7	+/– 3½ +/– 7(√3)/2
8	4(√3)	4	8	+/– 4 +/– 4√3
9	9(√3)/2	4 1/2	9	+/–4½ +/– 9(√3)/2

Another Interface Between the 90° Coordinate System and the 60° Coordinate System

In the 90° coordinate system the radius r of a circle is the hypotenuse of a right triangle. The perpendicular leg h extending down from the point (x_o, y_o) on the circle is the sine of the opposite angle θ, whereas, the base of the triangle is the cosine of the same angle, the angle θ which the radius makes with the x-axis. Let this circle enclose a hexagon such that the sides of the hexagon are the chords of the circle. The point P(x, y) where the radius r intersects the hexagon is the start of the 60° coordinate system. A line b, whose length is y, extending down from the point P(x, y) to intersect the x-axis at 60° is the y coordinate. From that point where line b intersects the x-axis back to the origin is the x coordinate. The lengths x and y of these two lines added together give the same length as the radius r.



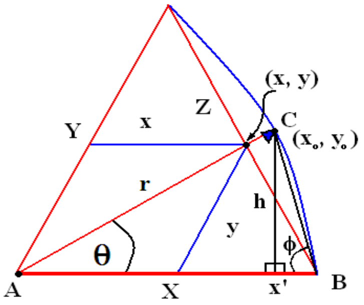
x' is the base of the right triangle in the 90° coordinate system.

r – x' is the distance between the two points (x_o, y_o) and (x, y). Why?
First, because X = r. That's a given.

Second, the right triangles Cx'B and C(x, y)B share the same side CB, the chord of the arc between C and B.

Third, the angle between r and CB and the angle between X and CB are equal, and

Fourth, angle Bx'C = angle C(x, y)B, they both being right angles, and since there are two angles in the two right triangles that are equal plus the fact that they share one side, the two triangles are equal.



Therefore, (x, y)C = x'B. A vertical dropped down from (x, y) gives you the imaginary number. y = h. Why?

First, the two right triangles are equal.

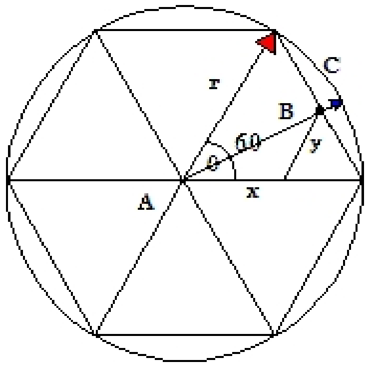
Second, the two bases of the right triangles are equal.

Third, the two right triangles share the same side CB.

Let the two equal bases be b. Let the side CD be a. Then h = $\surd(a^2 + b^2)$. But (x, y)B is also equal to $\surd(a^2 + b^2)$, and y = (x, y)B because (x, y)B is part of an equilateral triangle where y is one of the sides, so, h = y.

Now if h = sin θ , then y is also equal to sin θ. cos θ is merely found by subtracting y from r.

The point (x, y) is on the secant z, and not on the circle c. If the length of the secant approaches zero, then we might say that the point (x, y) coincides with a point on c, but this mathematics is more concerned with points on the hexagon.



Remember that x = r cos θ and y = r sin θ .

For a point extending from the side of a smaller hexagon within a larger hexagon to the side of that enclosing hexagon, and the two hexagons share a common center, then the outer point

$$(x_o + r \cos\theta, y_o + r \sin\theta)$$

is an extension of the inner point (x_o, y_o).

Chapter Five

Imaginary Numbers

Is Z an Imaginary Number?

In order to talk about 2-dimensional measurement, we must first talk about imaginary numbers. In Nature's way of measuring there is no imaginary numbers. Traditionally, $\sqrt{(-1)}$ is given the symbol i, and an imaginary number $z = a + ib$. (Some authors use j.) But the number $a + ib$ can be represented as the number $a + b$ or as the ordered pair (a, b) or (x, y) which is so similar to $z = x + y$ that all references to imaginary numbers $a + ib$ will be referred to from now on as $a + b$ or $x + y$, both of which is the imaginary number z or the coordinate z or a point (x, y) on the line c whose length is z. If x, y, and z are unit vectors, then $cz = ax + by$ or $c|_z = a|_x + b|_y$ (the z, x, y parts of a number) are also replacements of the imaginary number $z = a + ib$.

The logarithmic representation of $a + ib$ is $re^{i\theta}$. Since we are representing $a + ib$ as $a + b$, then $re^{i\theta}$ can be replaced with re^{θ}. When $\theta = \omega t$, then $a + b = re^{\omega t}$ and is a vector rotating in a counterclockwise rotation with an angular velocity of ω . For addition, $a + b$ is used, and for multiplication, re^{θ} is used so the exponents only need to be added.

The complex number $a + b$ can also be written as $(\cos\theta + \sin\theta)$ where $a = \cos\theta$ and $b = \sin\theta$. Also, $re^{\theta} = r(\cos\theta + \sin\theta)$.

The conjugate of z is $-y - x$, and the conjugate of $-z$ is $y + x$.

A complex number is defined as the endpoint of any vector, and a complex plane, any plane in which a vector is drawn from the origin out to the z-axis. If you plot a complex number or a vector in the complex plane (in other words, a plane in which a vector is drawn), then r will be the distance from the origin to the point on the z axis and θ will be the angle vector r makes with the x-axis.

DeMoivre's Formula[2]

DeMoivre's formula is the following:

$$(\mathrm{Cos}(\theta) + i\mathrm{Sin}(\theta))^n = \mathrm{Cos}(n\theta) + i\mathrm{Sin}(n\theta) \text{ where } 0^o \le \theta \le 90^o.$$

Using a variation of this formula, let us use

$$(\mathrm{Cos}(\theta) + \mathrm{Sin}(\theta))^n = \mathrm{Cos}(n\theta) + \mathrm{Sin}(n\theta) \text{ where } 0^o < \theta \le 60^o.$$

This formula is useful when you have a complex number and want to raise it to some power without doing a lot of work.

---

[2] Taken from Doctor Benway, The Math Forum at http://mathforum.org/dr.math/

Write the complex number re^{θ} as r (Cos(θ) + Sin(θ)) and raise it to a power n.

Essentially what you are doing is taking a complex number of the form

a + b, and

converting it to the form

re^{θ},

raising it to a power in that form, then converting back to the first form. Observe:

$$\begin{aligned}(r\mathrm{Cos}(\theta) + r\mathrm{Sin}(\theta))^{n} &= (r(\mathrm{Cos}(\theta) + \mathrm{Sin}(\theta))^{n} \\ &= (r^{n})(\mathrm{Cos}(\theta) + \mathrm{Sin}(\theta))^{n} \\ &= (r^{n})(e^{\theta})^{n} \\ &= (r^{n})(e^{n\theta}) \\ &= (r^{n})(\mathrm{Cos}(n\theta) + \mathrm{Sin}(n\theta))\end{aligned}$$

Of course knowing DeMoivre's formula allows us to go straight from

$$(r(\mathrm{Cos}(\theta) + \mathrm{Sin}(\theta))^{n}$$

to

$$(r^{n})(\mathrm{Cos}(n\theta) + \mathrm{Sin}(n\theta)).$$

Tetrahedral Roots of Numbers as Planes [3,4]

It will be found that the 3-dimensional manifold of a 60^{o} coordinate system can be obtained from a 90^{o} coordinate system using cubic roots and translated into tetrahedral roots of a system.

If $x^{3} = N$, where N is some expression (which could be a constant), then you have a third degree equation, so there must be three roots.

Suppose $z^{3} = 8$, z being a complex number. Now taking the tetrahedral root of each side (as if each edge of the tetrahedron with volume of 8 is divided into 2) you have z = 2, however, there are two other tetrahedral roots for this equation.

$$\text{Let } z^{3} = 8(1 + 0).$$

$$(\text{Remember that } 0^{o} \leq \theta \leq 60^{o})$$

But since Cos(6k) = 1 and Sin(6k) = 0 where k is any integer, we could write the equation as

$$z^{3} = 8(\mathrm{Cos}(6k) + \mathrm{Sin}(6k)).$$

---

[3] Taken from Doctor Anthony, The Math Forum at <http://mathforum.org/dr.math/>

[4] The trigonometry here is based upon the hexagon and a 60^{o} cycle.

Take the tetrahedral root of both sides, and use DeMoivre's theorem which shows that:

$$z = [8(\mathrm{Cos}(6k) + \mathrm{Sin}(6k))]^{1/3}$$
$$z = [8^{1/3}(\mathrm{Cos}(6/3k) + \mathrm{Sin}(6/3k))]$$
$$z = 2[\mathrm{Cos}(2k) + \mathrm{Sin}(2k)]$$

Letting k = 0, 1, 2,

$k = 0$ gives $z_1 = 2[(\mathrm{Cos}(0) + \mathrm{Sin}(0)]$
$= 2(1 + 0) \Rightarrow 2(1, 0)$
$= 2|_x$ (the one real root along the x-axis)
$k = ½$ gives $z_3 = 2(\mathrm{Cos}(1) + \mathrm{Sin}(1))$ [1 is 1/6th of a cycle of six.]
$= 2(0 + 1) \Rightarrow 2(0, 1)$
$= 2|_y$ (along the y-axis)
$k = 1$ gives $z_2 = 2(\mathrm{Cos}(2) + \mathrm{Sin}(2))$ [2 is 1/3rd of a cycle of six.]
$= 2(0 + 0) \Rightarrow 2(0, 0) \Rightarrow 2(0, 0, 1)$
$= 2|_z$ (along the z-axis because $x = y = 0$)

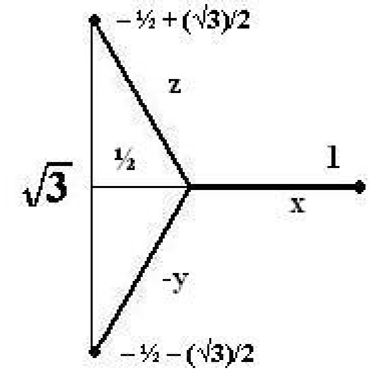
(Here, the accepted view is towards the observer, which would lead to a basis of vectors, but I use the view of away from the observer, so normally I would take this to be a – y.)

So, if $z^3 = 8$, we have the three roots of $2|_x$, $2|_z$ and $2|_y$, each 2 being on one of the three axes. If we give k more values, 3, 4, 5, ... we simply repeat the three roots already found.

Let's do this over again. First, we use one half of a unit equiangular, equilateral triangle. The height is (√3)/2, and the base is ½. Using the Pythagorean Theorem, the hypotenuse z is,

$$\sqrt{((1/2)^2 + (\sqrt{3})/2)^2)} = 1.$$

Using the 90º coordinate system, the z component is (–1/2, (√3)/2), and the y component is(–1/2, –(√3)/2) with the x component as 1. (Using the Pythagorean Theorem, we used the absolute values of the coordinates.)



So, if $z^3 = 1$, then $z^3 = [\mathrm{Cos}(6k) + \mathrm{Sin}(6k)]$
$z = [\mathrm{Cos}(6/3k) + \mathrm{Sin}(6/3k)]$
$z = [\mathrm{Cos}(2k) + \mathrm{Sin}(2k)]$

Using the above example for $z^3 = 8$, we see that when
k = 0 we have [1, 0],
k = ½ we have [0, 1], and when
k = 1 we have [0, 0] => [0, 0, 1].

These vectors refer to the basis vectors i, j, k of the 60° coordinate system. Refer back to pages 18-20 to see what I am talking about. Although I didn't mention the basis vectors, it can be conjectured from the previous information about the axes of the hexagonal coordinate system.

The three basis vectors [1, 0, 0], [0, 1, 0], and [0, 0, 1] , can be presented as three matrices: or one matrix:

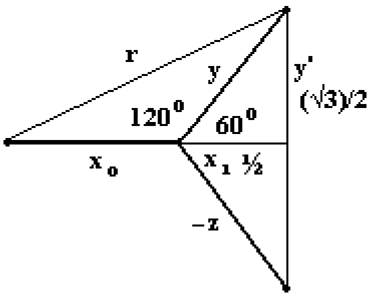
$$\underset{i}{\begin{bmatrix}1\\0\\0\end{bmatrix}} \quad \underset{j}{\begin{bmatrix}0\\1\\0\end{bmatrix}} \quad \underset{k}{\begin{bmatrix}0\\0\\1\end{bmatrix}} \qquad \text{or} \qquad \begin{bmatrix}1\,0\,0\\0\,1\,0\\0\,0\,1\end{bmatrix}$$

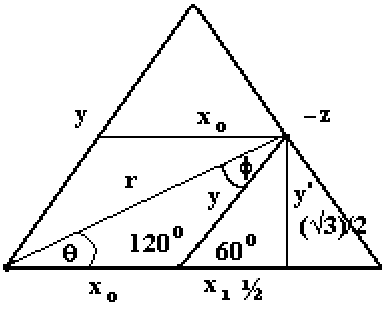
An imaginary number can be thought of as the endpoint of a vector with the origin at the origin of the coordinate system, and each vector can be presented as a linear combination of the basis vectors using matrices:

$$a \cdot \begin{bmatrix}1\\0\\0\end{bmatrix} + b \cdot \begin{bmatrix}0\\1\\0\end{bmatrix} + c \cdot \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}$$

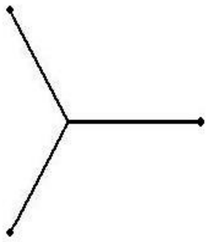
The Interface Between the 90^o Coordinate System and the 60^o Coordinate System

In the 90^o coordinate system sin 60^o = $(\surd 3)/2$ is the y coordinate of a solution of $z^3 = 1$ and is shown in the figure below as y'. The x coordinate is cos 60^o= ½ and is shown to the left as x_1. In the 60^o coordinate system, $y = \surd((\surd 3)/2)^2 + (1/2)^2)$ as shown in the figure to the right is the y coordinate and x_o is the x coordinate. In the 90^o coordinate system, the y-axis is perpendicular to the x-axis, but in the 60^o coordinate system, the y-axis is 60^o to the x-axis.

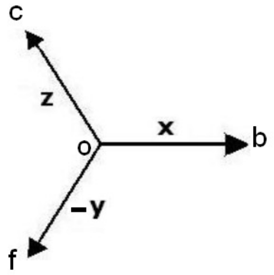




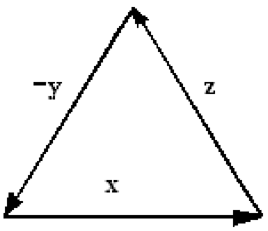
Let an equilateral triangle be drawn around the above figure such that the origin of the 90^o coordinate system is the midpoint of the base of the drawn triangle such that this base has the length $x_o + x_1$. Extending from this point is y, the left side of a smaller equilateral triangle, the base of which is twice the length of x_1. Call this the y-triangle. Extending from the top apex of the y-triangle is another equilateral triangle with a base equal to the length of x_o. Call this the x-triangle. Since the base of the x-triangle is parallel x_o , and the x and y triangles are equilateral triangles, then the right sides of the x and y triangles are collinear and add up to twice the length of – z, such that x_o, y, and – z form a basis for the 60^o coordinate system. The sides of the x and y triangles enclose a parallelogram in which a vector r is the diagonal. The x-triangle, the y-triangle, and the parallelogram between them becomes an equilateral triangle. The angle of r is θ and changes as r is rotated clockwise or counterclockwise. Likewise, the opposite angle ϕ also changes as r rotates. The 120° angle between θ and ϕ remains a constant as r rotates up or down because the adjacent 60° angle remains constant. The areas of the x and y-triangles change as r rotates such that the two areas added together remain constant as well as $x_o + y$ remains constant, the length $2|- z|$. Remembering that x_o, y, and – z are the complex values of the tetrahedral root of a number, and that a vector retains its original values during a translation, in other words, it can be moved anywhere as long as the length and angle remain constant, these 3 complex values as vectors can be translated to become the sides of an equilateral triangle. Doubling $(x_o, y, – z)$, these values become the length of the sides of the equilateral triangle encasing the above figure showing the solutions to $z^3 = 1$. In the above figure of the equilateral triangle, $2y = 2\sqrt{}(y'^2 + x_1^2)$ which is $2\sqrt{}((\sqrt{}3)/2)^2 + (1/2)^2) = 2$, $2| – z| = 2(x_o + y)$, and $x_o + 2x_1$ are in the 90° coordinate system, and $2x_o$, 2y, $2 | – z|$ are in the 60° coordinate system. The y coordinate is the key to this transition because of its relation to y' and x_1. Therefore, the solutions of $z^3 = 1$ become the basis for the 60° coordinate system.



If you represent the three roots of $z^3 = 1$ on a 90° coordinate system that has real values along the x axis and imaginary values along the y axis, the three roots will appear as the three spokes of a wheel, with the complex "z" values lying on a circle of a unit radius. One root will lie along the positive x axis, and the other two at +120° and – 120° to the x value on the x axis. So the roots are symmetrically spaced round the circle. In fact this is always the way that tetrahedral roots of a real number will look. If you take the tetrahedral root of an imaginary number, say i, then you still get three spokes, but they will be rotated to lie along the 60°, 180°, and the 300° lines on the unit circle. Still, each of the axes are 120° apart from each other.



becomes

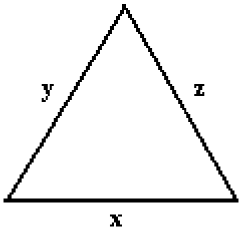


Therefore, the tetrahedral roots of a number can be represented as a triangle having three axes x, – y, and z having a counterclockwise rotation, which will be defined as a spinor later on. In fact, it is the smallest circuit one could have, especially smaller than a circle or a square.

A number tetrahedroned becomes a spinor

Using the tetrahedral roots we can create a 60° coordinate system. The tetrahedral roots of 1 gives us a unit triangle, then the roots of 2 and then 3, etc., give us a scale along the y and x axis with the z axis becoming longer and longer as it steps away from the origin (where the x and y axes touch). The tetrahedral roots of 1 become the basis for the 60° coordinate system. In other words, for the x, z and – y axis, the bases are $1|_x$, $1|_z$, and $-1|_{-y}$, and for the y, – x, and – z axes the bases become $1|_y$, $-1|_{-x}$, and $-1|_{-z}$.

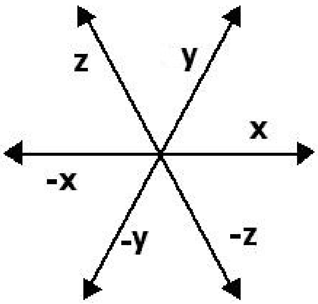
A spinor is the smallest area of a plane and is the smallest circuit. Any planar area can be formed of plus and minus spinors (counterclockwise and clockwise) which will be formally defined in a later chapter. Spinors form a vector space of linear independent vectors.



When it is desired to speak only of the positive space within the 60° coordinate system, we take the absolute value form of the x, y and z axes to form an equilateral triangle whose base vectors are linearly dependent. In other words, z = x + y. We will subsequently use the form z = x + y unless we are talking about spinors or bivectors. (It may be remembered that $\mathbf{z} = \mathbf{x} + \mathbf{y}$ is the equation of a triangle, and z = x + y gives the x and y coordinates.)

Any directional line segment or vector can be translated as long as it's angle with respect to some horizontal line remains constant. Let there be 3 vectors ob with length x, of with length y, and oc with length z such that they are separated by equal angles. If the length z is always equal to the lengths x and y, then x + y = z. These three vectors can be translated to form an equilateral, equiangular triangle in which all three internal angles are 60°.

Generalizing, let these vectors of x, – y, and z, axes be only half of the axes of a hexagon. They have the angles of 0°, 120° and 240°. Then the other axes y, – x, and – z are at 60°, 180° and 300° respectively. This system of vectors form a basis for and defines the 60° coordinate system. These axes form the axes of a hexagon. It becomes a projection of the x, y, z, 90° coordinate system onto an imaginary plane made up of six vectors, three positive and three negative. These vectors can be translated into two spinors, each going in opposite directions, the x, z, – y going in the counterclockwise direction, and the y, – x, – z going in the clockwise direction and being a conjugate of the first. The resultant direction will then be null, showing the resultant vectors of the coordinate system are null and static.



Chapter Six

A 60° Coordinate System

A Vector Space

Here is a definition of a vector space from http://dictionary.search.yahoo.com/search?p=vector%20space : A system consisting of a set of generalized vectors and a field of scalars, having the same rules for vector addition and scalar multiplication as physical vectors and scalars.

Either generally or specifically, a vector space is a mathematical structure that follows certain rules or axioms that define the addition of vectors and the scalar multiplication of vectors.

The vectors of the vector field defined within the boundaries of the VE are confined to the multiplication by the natural numbers 0, 1, 2, 3, …, n. Also, the length of each vector **v** is defined as $\mathbf{v} = \alpha\mathbf{u}$ where α is a scalar and **u** is a unit vector. The vector **v** can also be multiplied by a further scalar β such that $\mathbf{v}' = \beta\mathbf{v}$.

If **a** and **b** are non-zero vectors and $\mathbf{a}\beta = \mathbf{b}$, then and and be are not only collinear, but are linearly dependent.

Non-zero vectors $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$ are said to be linearly independent if

$$\mathbf{x}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) = \alpha_1\mathbf{a}_1 + \alpha_2\mathbf{a}_2 + \alpha_3\mathbf{a}_3 + \ldots + \alpha_n\mathbf{a}_n$$

is not zero for any combination of scalars $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$ (not all zero).

Let the VE be a vector space. Its size can be expanded by either adding a vector to each of the vectors within the VE or by multiplying each vector within the VE by a constant scalar. By these two operations the VE expands or contracts.

Taking any vector within the VE, pertaining to the four intersecting hexagons within the VE, and adding another vector to it or multiplying it by a scalar such that there is no distortion of the VE, we define that vector as independent of the vectors that make up the VE. If in the expansion or contraction of the VE there are other vectors within the VE that do not contribute to the structure of the VE then these extraneous vectors are either zero vectors, vectors in translation, or vectors added to or multiplied by some constant scalar which expands or contracts the VE. Also, they exist within one of the equilateral triangles in one of the four hexagonal planes within the VE or between any two intersecting hexagonal planes.

The resultant of the addition of n vectors is the longest diagonal of a generalized parallelepiped of n dimensions. The limit of the resultant is the boundary of the VE. The addition of two vectors or the multiplication of a vector by a scalar does not extend beyond the VE no matter the origin of the point within the VE. Although, the VE can be extended out as far as necessary to accommodate any length of a resultant vector. Therefore, the scalar that multiplies all the structural vectors of the VE produces a greater rate of increase than does the scalar that multiplies the independent vector. That goes for any vectors added.

Say within the VE there is a unit vector u projecting out from the center of the VE along each axis of the VE. By definition these vectors may be multiplied by a scalar such that each one of the vectors are multiplied by the same scalar a. In other words, if $v = \alpha_0 u_0 + \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_{12} u_{12}$ then each α_i is equal to the same number β. It is all or nothing so that the VE shrinks or expands evenly. Also, when all $\alpha_i = 0$, vector $v = \alpha_0 u_0 + \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_{12} u_{12} = 0$. That makes all u_i linearly independent. Each hexagonal plane within the VE is separated by 60º, and each axis within each hexagonal plane is separated by 60º. Therefore the vectors u_i are separated by the same angle. Add that to the fact that for any two vectors u_a and u_b, $u_a \cdot u_b = 0$ means that these vectors are orthogonal. Being linearly independent and orthogonal, the vectors u_i are called a basis $\{(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n\}$ for the vector space $\mathbf{x}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$ within the VE. There are 6 positive and 6 negative basis vectors.

The basis vectors for the VE can be enumerated as u_0 ,u_1 , u_2 , … ,u_{12} where twice the length of a unit basis vector is an axis of the VE such that $2|u_i| = v_{mn}$ and m = 1, 2, 3, 4 and n = 1, 2, 3.

A list of possible axes are $\begin{bmatrix} V_{11} V_{12} V_{13} \\ V_{21} V_{22} V_{23} \\ V_{31} V_{32} V_{33} \\ V_{41} V_{42} V_{43} \end{bmatrix}$ which form a matrix.

Each of the 4 planes within the VE have 6 basis vectors but 3 axes. It is an error to think then there are 4 x 6 = 24 basis vectors. 6 axes are shared with other planes as the planes intersect. If 2 planes intersect, there are not 2 x 3 = 6 axes, but 2 + 3 = 5. One is excluded. When 3 planes intersect, 1 + 2 = 3 axes are excluded and the axes add up to (3 x 3 = 9) – 3 = 6. So when 4 planes intersect, 1 + 2 + 3 = 6 axes are excluded, leaving (4 x 3 = 12) – 6 = 6.

(For each axis in the VE, there are two hexagonal planes passing through it. 4 planes x 3 axes = 12. But some of these axes are repeats. There are one of three, two of three and three of three. So, 3 + (3 – 1) + (3 – 2) + (3 – 3) = 6. But, there are 6 positive and 6 negative or 12 axes.)

The equation that deals with combinatorics is

$$C_x^n = (n!)/[x!(n-x)!],$$

so the combination of the indices of the axial matrix are 4 planes taken 2 basis vectors at a time:

$$C_2^4 = (4!)/[2!(4-2)!] = 6 \text{ combinations or 6 axes.}$$

Each axes has two basis vectors. That is, there are 6 positive and 6 negative basis vectors making up 12.

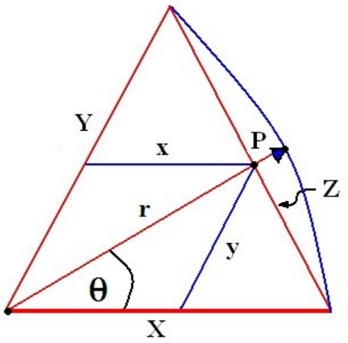
Each point in a VE is 12-dimensional. Also, each outer apex of the VE is 12 representing the center points of 12 spheres tangent to each other and touching a central sphere in 12 points.

Chapter Seven

A Geometric Algebra

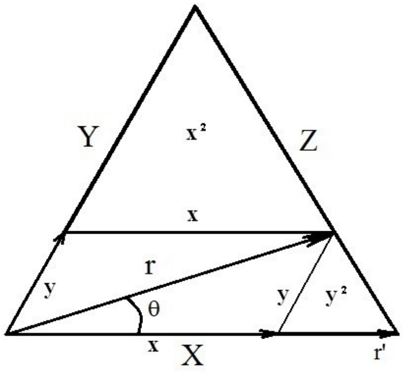
(inspired by David Orlin Hestenes)

Divisions of an Equilateral Triangle



I will be speaking of two kinds of divisions of an equilateral triangle. It is in reference to the division of one side of the triangle via a vector coming from the opposite corner.

The first division comes from a triangle with sides X, Y, and Z. A vector **r** collinear with Y and rooted at the XY apex such that the length of **r** = y = z = x is rotated down from the YZ apex to the ZX apex. The vector **r** cuts across the side Z as it rotates, dividing the triangle into two equilateral triangles and a parallelogram. The sides of the parallelogram x and y are also the sides of the two smaller triangles such that Z is divided into two line segments x and y. This can be done by any vector with the properties of **r** extending from any corner and sliding across the opposite side. What is remarkable is that the length z of **r** equals x + y. In other words, z = x + y.



The other division of the XYZ triangle comes from the vector addition of two vectors laid over the sides whose lengths are x and y such that

$$\mathbf{r} = \mathbf{x} + \mathbf{y}$$

which extends from a corner of the equilateral triangle up to but not extending beyond the opposite side, Z. In other words, **r** is the diagonal of the parallelogram having sides whose lengths x and y.

Note that it doesn't matter if the side whose length is x is above or below **r** or if the side whose length is y is to either side. The lengths x and y are the same coordinates of the point P on Z, where P is the end point of **r** or where **r'** crosses Z.

These two divisions made by **r'** and **r** of the XYZ triangle have a similar property. They can both be represented as γ = α + β. The only question is what does “+” mean? Obviously, the meaning of “+” has to be changed depending upon what you are adding. If “+” is previously defined, then γ = α + β has meaning. But still, the two divisions have similarity. The triangle is divided into three smaller spaces, two smaller triangles and a parallelogram.

Both **r'** or **r** cut Z at P(x, y) into line segments whose lengths are x and y, forming two inner triangles, the x triangle above the parallelogram and the y triangle below and to the right, with volumes of x^2 and y^2. These two triangles radiate from the endpoint P(x, y) of **r** and form the boundaries of the parallelogram with a volume of xy that surrounds **r**. The x triangle cuts Y into segments whose lengths are x and y, and the y triangle cuts X into segments whose lengths are x and y, and both triangles cut Z into segments whose lengths are x and y.

The Inner Product

Let an equilateral triangle be divided into two other equilateral triangles by a vector **r** traversing the space between one apex and its opposite side. This creates an x triangle above **r**, a y triangle on the right side, and a parallelogram separating the two smaller triangles such that the bottom side of the x triangle is the top side of the parallelogram, and the right side of the parallelogram is the left side of the y triangle. The diagonal of the parallelogram is the vector **r**. The base of the encompassing equilateral triangle is X and **r'** is collinear with and has the same length as X. The projection of a vector **r** onto another vector **r'** takes the form of the x part $\mathbf{r}_x$ of **r**. The left corner of the y triangle divides X into lengths x and y and takes away y from X to leave x. Now the projection of **r** onto **r'** is denoted by $\mathbf{r} \cdot \mathbf{r'} = |\mathbf{r}_x|$ and is called the dot product or inner product.

Since X = x + y, and x = $|\mathbf{r'}|$, then $|\mathbf{r'}| = x + y$ which is related to the inner product because $\mathbf{r} \cdot \mathbf{r'} = |\mathbf{r'}| - y = x$. This is true because a copy of **r** slides down the y side of the y triangle and lays over x which becomes the shadow of **r**. Since X = x + y, x is a line segment of X, and x is a scalar, in other words, the magnitude of the vector $\mathbf{r}_x$, the bottom side of the parallelogram separating the x and y triangles.

The x and y parts of **r** can be laid over **r'** and called place vectors, $\mathbf{r}_x$ and $\mathbf{r}^*_y$ (where $\mathbf{r}^*_y$ is $\mathbf{r}_y$ rotated 60°) such that $|\mathbf{r}_x| = x$ and $|\mathbf{r}^*_y| = y$ where the start point of both $\mathbf{r}_x$ and $\mathbf{r}^*_y$ is the start point of **r** and **r'**, $\mathbf{r}_y$ is the left side of both the parallelogram and the y triangle. So $\mathbf{r} \cdot \mathbf{r'} = |\mathbf{r}_x|$ which is the scalar x and $|\mathbf{r'}| = |\mathbf{r}_x| + |\mathbf{r}_y|$.

It must be remembered that $y = \sin\theta$ and $x = \cos\theta$ if $|\mathbf{r'}| = 1$. If $|\mathbf{r'}| > 1$, then

$$\mathbf{r} \cdot \mathbf{r'} = |\mathbf{r}| \cos\theta.$$

This assumes the definitions $1 \geq \cos\theta \geq 0$ and $0 \leq \sin\theta \leq 1$. This means that when $\cos\theta = 1$ then $\sin\theta = 0$ and when $\sin\theta = 1$ then $\cos\theta = 0$.

Note: If $|\mathbf{r'}| = z = 1$, then $y = \sin\theta$ and $x = \cos\theta$. **r'** or **r** cuts Z into x and y or cos θ and sin θ. Then the corresponding x and y triangles have sides equal to $x = \cos\theta$ and $y = \sin\theta$. For larger triangles, $ny = n \sin\theta$ and $nx = n \cos\theta$ for any integer n.

The traditional definition of the inner product is $\mathbf{r} \cdot \mathbf{r'} = |\mathbf{r}||\mathbf{r'}|\cos\theta$. It must be understood, though, that this is defined under a 90° coordinate system. As I have been saying, the 60° coordinate system simplifies, and as expected, the inner product is simply $|\mathbf{r}| \cos\theta$. Also, it has been seen that the multiplication of two vectors can simply be the addition of the length of one vector **r'** and the y part $\mathbf{r}_y$ of **r** going in the opposite direction, thus $|\mathbf{r'}| - |\mathbf{r}_y| = |\mathbf{r}_x| = |\mathbf{r}| \cos\theta$. Therefore, here is another simplification. The inner product is defined simply as:

$$\mathbf{r} \cdot \mathbf{r'} = |\mathbf{r'}| - |\mathbf{r}_y|.$$

Other definitions and simplifications of the inner product are:

$$\mathbf{r} \cdot \mathbf{r}' = |\mathbf{r}| \cos \theta,$$
$$\mathbf{r} \cdot \mathbf{r}' = |\mathbf{r}_x|$$
$$\mathbf{r} \cdot \mathbf{r}' = x \text{ (as a part of Z).}$$

The inner product has these properties:

- In Euclidean space, the inner product is always positive.
  $\mathbf{a} \cdot \mathbf{a} > 0$
- We define the cosine of the angle between the two vectors **a** and **b** as
  $\mathbf{a} \cdot \mathbf{b} / |\mathbf{a}| = \cos \theta$.
- Because we are talking about relative direction in the definition of $\mathbf{a} \cdot \mathbf{b}$, the angle between **a** and **b** remains constant and produces an important symmetry property as
  $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.
- The relation to scalar multiplication of vectors is expressed by
  $(\mathbf{a}\boldsymbol{\lambda}) \cdot \mathbf{b} \Rightarrow \boldsymbol{\lambda}(\mathbf{a} \cdot \mathbf{b}) \Rightarrow \mathbf{a} \cdot (\mathbf{b}\boldsymbol{\lambda})$ where $|\mathbf{a}|/|\mathbf{b}|$ remains constant and $\boldsymbol{\lambda}$ is positive, negative or zero. Meaning, that if **b** expands, then **a** expands at the same rate and visa versa.
- Its relation to vector addition can be expressed by the distributive rule:
  $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
- The magnitude of a vector is related to the inner product by
  $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 > 0$ ($\mathbf{a} \cdot \mathbf{a} = \mathbf{0}$ if and only if $\mathbf{a} = \mathbf{0}$.)
- If $\mathbf{a} \cdot \mathbf{b} = 0$ then **a** and **b** are orthogonal.
- If $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\cos \theta$ and $2(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a}$ then $2(\mathbf{a} \cdot \mathbf{b}) = (|\mathbf{a}|+|\mathbf{b}|)\cos \theta$.

The inner or dot product as matrices is A^TA

First of all, using matrices, row times column is a dot product. Given matrix Q, for the transpose of Q, Q^T, each orthonormal, independent row, q_i^T times each orthonormal, independent column q_i of matrix Q is I, the Identity matrix. Q is orthogonal if it is square. If Q is square, then $Q^TQ = I$ and therefore, $Q^T = Q^{-1}$, the inverse of Q. (I was told that this is true for permutations of Q also.)

To make a column of matrix A orthonormal, divide it by its length. Use the Pythagorean Theorem, where each square is an entry in the column, for a 90° coordinate system, but in a hexagonal coordinate system, remember that the length of any column z is simply $x + y$.

The projection P onto the column space of Q is $P = Q(Q^TQ)^{-1}Q^T = Q^TQ$. If Q is square, then $P = I$, the identity matrix.

P is always symmetric, and a projection twice is always a projection, vis. $(Q^TQ)(Q^TQ) = Q^TQ$.

For square matrices, the idea is to make them triangular.

If $A^TAx = A^Tb$, and $A = Q$, then $Q^TQx = Q^Tb$, and $x = Q^Tb$, where x and b are vectors.

What this means is that $x_i = q^T_ib$, which is an important equation in much of mathematics.

Suppose we have any two vectors **a** and **b** with an angle less than 60° between them. What we want is the orthogonal parts A and B of these vectors. What we want to do is to find the y part of **b** if $\mathbf{b} = b_x + b_y$; **a** is already coplanar with the x axis, and b_x is the projection of **b** onto **a**.

If A and B are the orthogonal vectors then $q_1 = A/||A||$ and $q_2 = B/||B||$ are the orthonormal vectors. A = **a**, and B = **b** – $(A^Tb) A/(A^TA)$.

To show A and B are perpendicular, A^TB, we show this equals

$$A^TB = A^T(\mathbf{b} - (A^T\mathbf{b}) A/(A^TA)) = 0.$$

For another orthogonal vector C, we have $C = \mathbf{c} - (A^T\mathbf{c}) A/(A^TA)) - (B^T\mathbf{c}) B/(B^TB))$. This shows that A, B, and C are orthogonal to each other, where $q_3 = C/||C||$.

The column space of matrix A is the same as the column space of QR or a linear combination of Q. R turns out to be an upper triangular matrix. If A = [a b], (two orthogonal columns) then $Q = [q_1\ q_2]$ (two orthonormal columns) and $R = [a_1{}^Tq_1,\ a_1{}^Tq_2; –, –]$ This represents a 2x2 upper triangular matrix, where $a_1{}^Tq_2 = 0$ (orthogonality), showing only the first column of the matrix.

The Properties of Vectors

The dimensionality of vectors usually follows orthogonality within a vector field $v(\alpha_1, \alpha_2, \ldots, \alpha_n)$ where any two vectors $\mathbf{v_i}$ and $\mathbf{v_j}$ are such that $\mathbf{v_i} \cdot \mathbf{v_j} = 0$. Different spaces are usually described as one dimensional, two dimensional, three dimensional, etc. or geometrically as points, lines, planes, volumes, etc. but there has been no traditional way to consistently describe a geometric property which has direction as the dimensions get higher. A line segment which has direction is called a vector and has these properties:

Given three vectors **a**, **b** and **c**:

A. Rules for vector addition

- **Equality :**
  a = **b**, iff **a** and **b** have the same magnitude and direction
- **Closure:**
  a + **b** = **c**, which is also the equation of a triangle
- **Commutative:**
  a + **b** = **b** + **a**
- **Associative:**
  (**a** + **b**) + **c** = **a** + (**b** + **c**)
- **Additive Inverse and zero vector:**
  a + (– **a)** = **0**, that is, **a** is rotated 180°, called the negative of **a**
  a + **0** = **a**
- **Subtraction:**
  a – **b** = **a** + (– **b**), the definition of subtraction as the addition of a negative

Example:

Using the rules for the inner product and for vectors, a common formula in trigonometry can be created. If **a** + **b** = **c**, each vector can be triangled by using the inner product:

$$\mathbf{c} \cdot \mathbf{c} = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = \mathbf{a} \cdot (\mathbf{a} + \mathbf{b}) + \mathbf{b} \cdot (\mathbf{a} + \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} \text{ which}$$

$$|\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2(\mathbf{a} \cdot \mathbf{b}) = |\mathbf{a}|^2 + |\mathbf{b}|^2 + (|\mathbf{a}|+|\mathbf{b}|)\cos\theta.$$

Since a = |**a**|, b = |**b**|, c = |**c**|, and **a · b** = **–** a cos C, this formula can be expressed in terms of scalars. So we have

$$c^2 = a^2 + b^2 - c \cos C$$

which is the **law of cosines**.

If C = 0, we have the Pythagorean Theorem. Of course, this needs some work, as there is no need for the Pythagorean Theorem in a hexagonal coordinate system.

B. Rules for multiplication for scalars (Greek letters)

- **Additive and multiplicative identities:**
  - (1)**a** = **a**
  - (0)**a** = 0, a vector with dimension zero
  - (– 1)**a** = – **a**, iff $\lambda = -1$, definition of scalar multiplication with a negative
- **Distributive:**
  - $\alpha(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{a} + \alpha\mathbf{b}$
  - $(\alpha + \beta)\mathbf{a} = \alpha\mathbf{a} + \beta\mathbf{a}$
- **Associative:**
  - $\alpha(\beta\mathbf{a}) = (\alpha\beta)\mathbf{a}$
- **Commutative:**
  - $\alpha\mathbf{a} = \mathbf{a}\alpha$
- **Magnitude and direction:**
  - if a scalar α = |**a**|, then **a** = |**a**|**â**, where **â** = **1**, the unite vector (and if a ≠ 0)
- **Collinearity:**
  - If **a** and **b** are non-zero vectors and $\mathbf{a}\beta = \mathbf{b}$, then and be are not only collinear, but are linearly dependent.
- **Linear independence:**
  - Non-zero vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are said to be linearly independent if

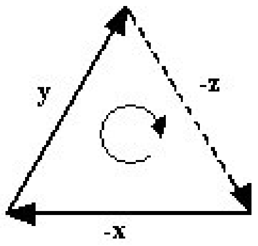
$$\mathbf{a}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) = \alpha_1\mathbf{a}_1 + \alpha_2\mathbf{a}_2 + \alpha_3\mathbf{a}_3 + \ldots + \alpha_n\mathbf{a}_n = 0.$$

C. Parametric Equations

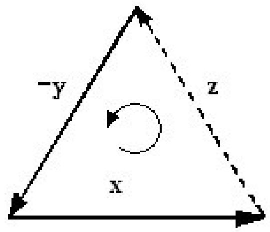
- **Line:** $\mathbf{x}(\alpha) = \alpha\mathbf{a} + \mathbf{b}$
- **Line segment** for $0 \leq \alpha \leq 1$

The Outer Product

Just as vectors were invented to define a directed line segment, so the bivector was invented to define the next higher relationship of dimensionality, the directed plane segment. This directed plane segment can be understood as a directed area $\mathbf{a}^2$ = **A** = A**A^**, where **A^** is the direction of the area A of an equilateral triangle and A = |**A**|, the magnitude of **A**. Although a bivector was defined as a parallelogram to be the product of two vectors, that is not the smallest segment of a plane. No, the equilateral equiangular triangle plays that role. (In some other geometry the circle may be the primary or the smallest segment of a plane, but the rectangle or square never can be because they are divisible by the triangle.)



Bivector with clockwise rotation



Bivector with counter clockwise rotation

Note: this is a bivector that is also a spinor. There are other bivectors which are not spinors.

As one vector sweeps across an angular area, it creates a triangular area. But what we are concerned with here is the product of two vectors. We know that $|\mathbf{a}|^2$ is the volume of a triangle, but, if we have $|\mathbf{a}| = |\mathbf{b}|$, as in an equilateral triangle, then $|\mathbf{a}||\mathbf{b}|$ is also an area of that triangle. The cross product $\mathbf{a} \mathbf{x} \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin \theta$ has been traditionally the defining area expressed as a vector perpendicular to the plane, representing the area of the plane, but it has the weakness of not being able to exist within the plane. But the bivector, having the same definition of area, $|\mathbf{a}||\mathbf{b}| \sin \theta$, where $0 < \sin \theta < 1$ and $\sin 60^\circ = 1$, and thus, $|\mathbf{a}||\mathbf{b}| = \mathbf{a} \wedge \mathbf{b}$ exists within the same plane as **a** and **b** and represents a directed plane. As **a** crosses into **b**, we say that $\mathbf{a} \wedge \mathbf{b}$, pronounced a "wedge" b, and is a directed area segment called a bivector. It could also be called a 2-vector. If a vector is a one vector, we can call a scalar a 0-vector, a vector, a 1-vector, and bivector, a 2-vector. I will describe down the line that a triangle constructed with three vectors coming from the solution of a third power equation, such as u^3, is a 3-vector or trivector, and a tetrahedron as a 4-vector. (It has rotation about the 4th vector.)

Just as the volume of an equilateral triangle whose equation is $z = x + y$, is z^2, the volume of a bivector (also an equilateral triangle) is $z^2 = (xy)^2$, where $\mathbf{x} = \mathbf{y}$ and $xy = \mathbf{x} \wedge \mathbf{y}$.
Only if $\mathbf{x} \neq \mathbf{y}$, then $\mathbf{x} \wedge \mathbf{y} = xy \sin \theta$.

Properties of a bivector:

1. The outer product of two vectors is **antisymmetric** and **anticommunitive**,
   $\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$, also,
   If $\mathbf{a} \wedge \mathbf{b} = \mathbf{B}$, then $-\mathbf{b} \wedge \mathbf{a} = -\mathbf{B}$.
   This follows from the geometric definition. Reversing the order of the vectors reverses the order of the bivector.
2. There is a correspondence between vector orientation and bivector orientation:
   $\mathbf{b} \wedge \mathbf{a} = \mathbf{a} \wedge (-\mathbf{b}) = (-\mathbf{b}) \wedge (-\mathbf{a}) = (-\mathbf{a}) \wedge \mathbf{b}$
   When the left side is switched to become the right side, a change in sign

occurs. Not so with the right side. Switch the right side with the left side, and there is no change in sign.

3. The outer product is **distributive**. That is,
    - **Right Distributive:**
      $(\mathbf{b} + \mathbf{c}) \wedge \mathbf{a} = \mathbf{b} \wedge \mathbf{a} + \mathbf{c} \wedge \mathbf{a}$
    - **Left Distributive:**
      $\mathbf{a} \wedge (\mathbf{b} + \mathbf{c}) = \mathbf{a} \wedge \mathbf{b} + \mathbf{a} \wedge \mathbf{c}$
4. The relationship between the magnitudes of vectors and the magnitude of bivectors is
   $|\mathbf{B}| = |\mathbf{a} \wedge \mathbf{b}| = |\mathbf{b} \wedge \mathbf{a}| = \frac{1}{2} |\mathbf{a}||\mathbf{b}| \sin \theta$
   The $\sin \theta$ is not part of the definition, since $\sin 60^\circ = 1$. It only is a comparison with trigonometry. The actual volume of a bivector/plane segment is ½ $|\mathbf{a}||\mathbf{b}|$ or ½ of a rectangle or parallelogram.
5. For bivectors **B** and **C**, and scalar λ, $\mathbf{C} = \lambda\mathbf{B}$ means that the magnitude of **B** is dilated by the magnitude of λ, that is, $|\mathbf{C}| = |\lambda||\mathbf{B}|$, and the direction of **C** is the same as **B**. If λ is the unit vector, positive or negative, then $(1)\mathbf{B} = \mathbf{B}$ or $(-1)\,\mathbf{B} = -\mathbf{B}$. Bivectors which are multiples of each other are said to be codirectional or of the same direction.
6. Dilating one side of the triangle dilates the other side also:
   $\lambda(\mathbf{a} \wedge \mathbf{b}) = (\lambda\mathbf{a}) \wedge \mathbf{b} = \mathbf{a} \wedge (\lambda\mathbf{b})$
   This is so with equilateral triangles.
7. $\mathbf{a} \wedge \mathbf{b} = 0$ iff $\mathbf{b} = \lambda\mathbf{a}$. If $\lambda \neq 0$, then $\mathbf{a} \wedge \mathbf{a} = 0$ which is shown by $\mathbf{a} \wedge \mathbf{a} = -\mathbf{a} \wedge \mathbf{a}$.
8. $\mathbf{a} \wedge \mathbf{b} = 0$ shows that **a** and **b** are parallel.
9. Bivectors form a linear space the same as vectors do.

"Given any non-zero vector **a** in the plane of bivector **B**, one can find a vector **b** such that
$\mathbf{B} = \mathbf{ba} = -\mathbf{ab}$,
$\mathbf{B}^2 = -\mathbf{B}^2 = -\mathbf{a}^2\,\mathbf{b}^2$,
$\mathbf{aB} = -\mathbf{Ba}$, that is, **B** anticommutes with every vector in the plane of **B**.
Every vector a has a multiplicative inverse: $\mathbf{a} -\mathbf{1} = \mathbf{1a} = \mathbf{aa}^2$
that is, geometric algebra makes it possible to divide by vectors."

David Hestenes

Commute or Anticommute?

If $\mathbf{a} \cdot \mathbf{b} = \mathbf{0}$ then $\mathbf{ab} = -\mathbf{ba}$. Orthogonal vectors anticommute, but because if $\mathbf{a} = \lambda\mathbf{b}$ then $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$ implies $\mathbf{ab} = \mathbf{ba}$ shows that collinear vectors commute.

Now, there is really no unique dependence on **a** and **b**. If $\mathbf{a}' = \mathbf{a} + \lambda\mathbf{b}$ we still have $\mathbf{a}' \wedge \mathbf{b} = \mathbf{a} \wedge \mathbf{b}$. Let $\lambda\mathbf{b} = \mathbf{c}$. Therefore, we have $(\mathbf{a} + \mathbf{c}) \wedge \mathbf{b} = \mathbf{a} \wedge \mathbf{b} + \mathbf{c} \wedge \mathbf{b}$ where $\mathbf{c} \wedge \mathbf{b} = \lambda\mathbf{b} \wedge \mathbf{b} = 0$, since $\lambda\mathbf{b}$ and **b** are parallel.

Example of Using the Properties of a Bivector:

The inner and outer products complement each another. Relations which are difficult or impossible to obtain with one may be easy to obtain with the other. The equation $\mathbf{a} \cdot \mathbf{b} = 0$ provides an expression of the perpendicular, whereas $\mathbf{a} \wedge \mathbf{b} = 0$ provides an expression of the parallel. Take the equation for the triangle, $\mathbf{a} + \mathbf{b} = \mathbf{c}$, where $\mathbf{a} = \mathbf{b} = \mathbf{c}$, in which each vector is wedged with the vector (a, b, c).

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \wedge \mathbf{a} + \begin{pmatrix} a \\ b \\ c \end{pmatrix} \wedge \mathbf{b} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \wedge \mathbf{c}$$

This can be expressed as

$$\boldsymbol{a} \wedge \boldsymbol{a} + \mathbf{a} \wedge \mathbf{b} = \mathbf{a} \wedge \mathbf{c}$$

$$\mathbf{b} \wedge \mathbf{a} + \boldsymbol{b} \wedge \boldsymbol{b} = \mathbf{b} \wedge \mathbf{c}$$

$$\mathbf{c} \wedge \mathbf{a} + \mathbf{c} \wedge \mathbf{b} = \boldsymbol{c} \wedge \boldsymbol{c}$$

Since $\boldsymbol{a} \wedge \boldsymbol{a} = \boldsymbol{b} \wedge \boldsymbol{b} = \boldsymbol{c} \wedge \boldsymbol{c} = 0$, this turns into

$$\mathbf{a} \wedge \mathbf{b} = \mathbf{a} \wedge \mathbf{c}$$

$$\mathbf{b} \wedge \mathbf{a} = \mathbf{b} \wedge \mathbf{c}$$

$$\mathbf{c} \wedge \mathbf{a} + \mathbf{c} \wedge \mathbf{b} = 0$$

Only two of these equations are independent, the last equation being the sum of the first two. We can write the first two equations on a single line, reversing the second equation:

$$\mathbf{a} \wedge \mathbf{b} = \mathbf{a} \wedge \mathbf{c} = \mathbf{a} \wedge \mathbf{b} = \mathbf{c} \wedge \mathbf{b}$$

But since $\mathbf{a} \wedge \mathbf{c}$ is equated with $\mathbf{a} \wedge \mathbf{b}$ twice,

$$\mathbf{a} \wedge \mathbf{c} = \mathbf{a} \wedge \mathbf{b} = \mathbf{c} \wedge \mathbf{b}$$

Here are three different ways of expressing the same bivector as a product of vectors. This gives three different ways to express its magnitude:

$$|\mathbf{a} \wedge \mathbf{c}| = |\mathbf{a} \wedge \mathbf{b}| = |\mathbf{c} \wedge \mathbf{b}|$$

Using the scalar labels for the triangle and dividing by $|\mathbf{a}||\mathbf{b}||\mathbf{c}|$, we get:

$$(|\mathbf{a}||\mathbf{c}| \sin \theta_{\mathbf{b}}) / (|\mathbf{a}||\mathbf{b}||\mathbf{c}|) = (|\mathbf{a}||\mathbf{b}| \sin \theta_{\mathbf{c}}) / (|\mathbf{a}||\mathbf{b}||\mathbf{c}|) = (|\mathbf{c}||\mathbf{b}| \sin \theta_{\mathbf{a}}) / (|\mathbf{a}||\mathbf{b}||\mathbf{c}|)$$

which reduces to:

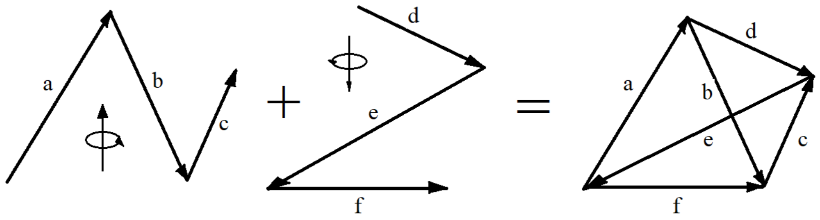
$$(\sin \theta_{\mathbf{b}}) / |\mathbf{b}| = (\sin \theta_{\mathbf{c}}) / |\mathbf{c}| = (\sin \theta_{\mathbf{a}}) / |\mathbf{a}| \text{ or}$$

$$(\sin \mathrm{a}) / \mathrm{a} = (\sin \mathrm{b}) / \mathrm{b} = (\sin \mathrm{c}) / \mathrm{c}$$

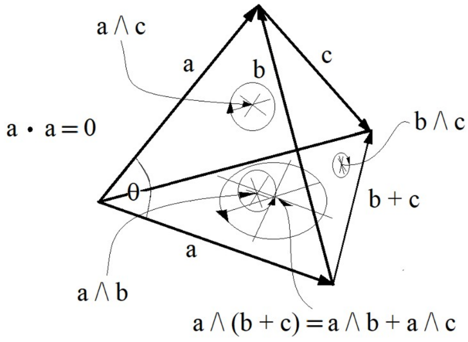
which is the **law of sines**.

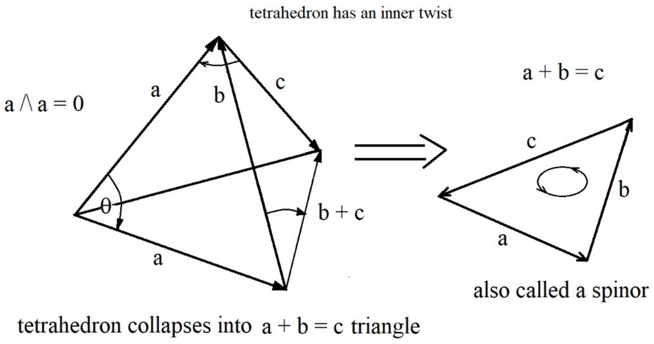
Spirals

Two spirals fit together to make a tetrahedron.



In working with vectors and geometry, one of the important dynamic systems is that of the spiral. Shown above is a vector representation of spirals, two going in opposite directions. When connected, they form a tetrahedron. Since the spiral is intrinsic to the tetrahedron, and the tetrahedron is a representation of four dimensions, the tetrahedron can twist and collapse into a triangular plane representing three dimensions. This is defined as a spinor, the smallest circuit. Let the below triangle represent the distributive property of bivectors. As **c** is rotated into **a** and **b** is rotated into **c** simultaneously, and as a sweeps across **b**, the intrinsic twist of the tetrahedron collapses it into a spinor.





The Spinor

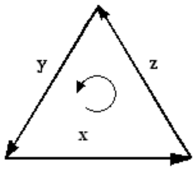
The spinor is defined when within a Cartesian coordinate system three unit vectors are the resultant of finding the cubed root of a function f^3 where one root (0, +1) is on the positive x-axis, another root (–1, +1) is found 120° counterclockwise in the second quadrant, and the third root (–1, –1) a 120° counterclockwise in the third quadrant. These three vectors can be translated such that the length and angles are invariant, forming an equilateral triangle.

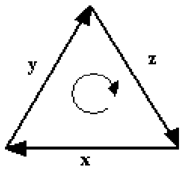
Another approach is when the vector **r** cuts the side Z of an equilateral triangle into two line lengths x and y such that Z => x + y. This divides the equilateral triangle into two triangles, the y triangle, $\mathbf{y}_x + \mathbf{y}_y = \mathbf{y}_z$, and the x triangle, $\mathbf{x}_x + \mathbf{x}_y = \mathbf{x}_z$ and a parallelogram. For an equilateral triangle, the principle of adding an x component and a y component produces a z component, whether you are talking about line segments or vectors. This process produces a linear space of dependent vectors. If it happens that ξ_y is flipped or rotated 180° such that $\xi_x + \xi_z = -\xi_y$, then ξ_y, ξ_x, and ξ_z are the result of a 3rd degree equation, and ξ_y, ξ_x, and ξ_z produce a linear space of independent vectors. The resultant equilateral triangle of vectors is called a spinor, replaces the tensor, and is the smallest circuit.

Trivectors

Let $\mathbf{S} = (\mathbf{x} \wedge \mathbf{y}) \wedge \mathbf{z}$ denote a trivector. This is a generalization of a **spinor**. Recall that in tetrahedral roots of a number, there were two solutions as equilateral triangles made up of vectors. One triangle has the vectors oriented in the clockwise direction and the other one in the counterclockwise direction. So $(\mathbf{x} \wedge \mathbf{y}) \wedge \mathbf{z}$ denotes a clockwise rotation, and $-(\mathbf{y} \wedge \mathbf{x}) \wedge \mathbf{z}$ denotes the opposite rotation. The idea that $\mathbf{z}$ is within the same plane as $\mathbf{x}$ and $\mathbf{y}$ is expressed by

$$(\mathbf{x} \wedge \mathbf{y}) \wedge \mathbf{z} = 0.$$





$-(\mathbf{y} \wedge \mathbf{x}) \wedge \mathbf{z}$ or $(\mathbf{x} \wedge \mathbf{y}) \wedge \mathbf{z}$

Example: Within a segment of a plane, let there be a region R such that R is divided into a finite number of consecutive spinors. Adding up the lengths of all the perimeters of the spinors, an equal number of clockwise spinors and counterclockwise spinors, defines the length of the boundary of R.
Another Example: Let there be a surface S such that R is a cross section of S, and let S be divided into a finite number of spinors. The addition of all the spinors on that surface sums to zero due to there being both an equal number of clockwise spinors and counterclockwise spinors.

The outer product can be generalized. Just as a plane segment is swept out by a rotating vector or directed line segment, a space segment is swept out by a rotating directed plane segment or bivector $\mathbf{a} \wedge \mathbf{b}$, moving at a distance and direction symbolized by the vector $\mathbf{c}$ producing a half tetrahedron. This is called a trivector. When two trivectors spin together and interlock, they form a tetrahedron.

The algebra of Hamiltonian quaternions contains 4 elements $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, but only three of these specify a vector. This can be generalized by defining the unit as a fourth vector. $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is generalized as $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$. This fourth dimension $\mathbf{d}$ can be interpreted as time, so the Hamiltonian can be demonstrated with a tetrahedron T^S, a double spiral or trivector.

We write the outer product of a bivector $\mathbf{a} \wedge \mathbf{b}$ with a vector c as $(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = T^R$.
As for bivectors, trivectors obey the associative rule:

$$(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}).$$

This rule can be determined from another rule: $(\mathbf{a} \wedge \mathbf{b}) = -(\mathbf{b} \wedge \mathbf{a})$ such that

$$(\mathbf{b} \wedge \mathbf{a}) \wedge \mathbf{c} = (-\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = -T^{R}.$$

Thus, the orientation of a trivector can be reversed by reversing the orientation of only one of its components. This makes it possible to rearrange the vectors to get

$$(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = (\mathbf{b} \wedge \mathbf{c}) \wedge \mathbf{a} = (\mathbf{c} \wedge \mathbf{a}) \wedge \mathbf{b},$$

which means that $(\mathbf{a} \wedge \mathbf{b})$ sweeping along $\mathbf{c}$, $(\mathbf{b} \wedge \mathbf{c})$ sweeping along $\mathbf{a}$, and $(\mathbf{c} \wedge \mathbf{a})$ sweeping along $\mathbf{b}$, all results in the same tetrahedron. But if

$$(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 0,$$

then $\mathbf{c}$ lies in the same plane as $\mathbf{a}$ and $\mathbf{b}$, and a tetrahedron is not produced.

Also, as bivectors are anticommutative, so are trivectors:

$$\mathbf{c} \wedge \mathbf{b} \wedge \mathbf{a} = -\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}.$$

Without such algebraic apparatus, any geometrical idea of relative orientation would be difficult to express. Adding more dimensions does not add any new insights into the relation between algebra and geometry. The displacement of a trivector $a \wedge b \wedge c$ along a fourth vector d does not produce a fourth-dimensional space segment analogous to a three dimensional tetrahedron, especially since the tetrahedron is enough to demonstrate a four dimensional manifold. The tetrahedron expresses space-time and is the final multiplicity for real dimensions.

So $(a \wedge b \wedge c) \wedge d = a \wedge b \wedge c \wedge d = 0$.

The Geometric Product

So far, we have a symmetric product $\mathbf{a} \cdot \mathbf{b}$ and an antisymmetric product $\mathbf{a} \wedge \mathbf{b}$. These quantities can be tied to a geometric manifold to help explain many mathematical properties. Geometric Algebra is Clifford's algebra, and Clifford decided to combine these two products into a geometric product

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}.$$

This may seem strange to combine a scalar with a directed area, but if you look at it as a complex number with a real part and an imaginary part, it is understandable.

Since $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$, and $\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$, by the symmetry/antisymmetry use of the terms, $\mathbf{ba} = \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \wedge \mathbf{a} = \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \wedge \mathbf{b}$.

The geometric product can be split into $\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}(\mathbf{ab} + \mathbf{ba})$ and $\mathbf{a} \wedge \mathbf{b} = \frac{1}{2}(\mathbf{ab} - \mathbf{ba})$. Added together as $\mathbf{ab} = \frac{1}{2}[(\mathbf{ab} + \mathbf{ba}) + (\mathbf{ab} - \mathbf{ba})]$, we can form other products in terms of the geometric product which can prove useful.

Properties of the Geometric Product:

1. General elements of a Geometric Algebra are called multivectors and are usually written in upper case, (**A, B, C**, …). These form a linear space in which scalars can be added to bivectors, and vectors, etc.
2. The geometric product is **associative**:
   $$\mathbf{A(BC) = (AB)C = ABC}$$
3. The geometric product is **distributive**:
   - Right distributive:
     $$\mathbf{(B+C)A = BA + CA}$$
   - Left distributive:
     $$\mathbf{A(B+C) = AB + AC}$$

   (Matrix multiplication is a good thing to keep in mind.)
4. Euclidean metric:
   $\mathbf{a}^2 = a^2$
   The triangle of any vector is a scalar. a = |**a**| is a positive scalar (a real number). The proof of this last property is to prove that the inner product of any two vectors is a scalar.
   $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 > 0$

Take the equation for a triangle, $\mathbf{c} = \mathbf{a} + \mathbf{b}$, and therefore, $\mathbf{c}^2 = (\mathbf{a} + \mathbf{b})^2$. Expanding,

$$\begin{aligned}(\mathbf{a} + \mathbf{b})^2 &= (\mathbf{a} + \mathbf{b})(\mathbf{a} + \mathbf{b}) \\ &= \mathbf{a}^2 + \mathbf{b}^2 + \mathbf{ab} + \mathbf{ba}\end{aligned}$$

It follows that

$$\mathbf{ab} + \mathbf{ba} = (\mathbf{a} + \mathbf{b})^2 - \mathbf{a}^2 - \mathbf{b}^2$$

In geometric algebra, $\mathbf{ab} = \mathbf{C}$ has the solution $\mathbf{b} = \mathbf{a}^{-1}\mathbf{C}$. Neither the inner product nor the outer product are capable of this inversion on their own.

Blades

In Geometric Algebra there are four different directional properties, a scalar is called a 0-vector, a vector is a 1-vector, a plane is a 2-vector, a space is a 3-vector, and a space-time is a 4-vector. These are different grades within the GA. We have seen different ways vectors can be multiplied, such as scalar, inner and outer products to express the relations between the different elements. The inner and outer products have been reduced to a single geometric product $\mathbf{ab}$, the addition of two different grades. If we generalize this idea, then the result is a multivector $\mathbf{A}$. $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 + \mathbf{A}_4$. $\mathbf{A}_k$, such that k = 0, 1, 2, 3, 4, is called the kth element of $\mathbf{A}$. This is called a k-vector or k-blade. It is also simply called a blade.

The ideas of geometric product $\mathbf{ab}$ and blades $\mathbf{A}_k$ can be brought together in

$$\mathbf{aA}_k = \mathbf{a} \cdot \mathbf{A}_k + \mathbf{a} \wedge \mathbf{A}_k.$$

Applying the associative rule for the geometric product:

$$\text{if } \mathbf{a(bc)} = \mathbf{(ab)c}, \text{ then}$$

$$\mathbf{a(b \cdot c + b \wedge c)} = \mathbf{(a \cdot b + a \wedge b)c}.$$

Then applying the distributive rule:

$$\mathbf{a(b \cdot c) + a \cdot (b \wedge c) + a \wedge (b \wedge c) = (a \cdot b)c + (a \wedge b) \cdot c + (a \wedge b) \wedge c}.$$

Now $\mathbf{a(b \cdot c)}$ and $\mathbf{a \cdot (b \wedge c)}$ are vectors, whereas, $\mathbf{a \wedge (b \wedge c)}$ and $\mathbf{(a \wedge b) \wedge c}$ are trivectors. Since vectors are different from trivectors, we can separately equate vector and trivector parts on each side of the equation.

$$\mathbf{a \wedge (b \wedge c) = (a \wedge b) \wedge c}$$

describes the associative rule, while

$$\mathbf{a(b \cdot c) + a \cdot (b \wedge c) = (a \cdot b)c + (a \wedge b) \cdot c}$$

applys the distributive rule. This is an easier way to derive the associative rule.

Parts of the same grade within the generalized equation for the geometric product are separately equated. Thus, if $\mathbf{a}$ is a vector and $\mathbf{B}_k$ and $\mathbf{C}_k$ are k-blades, then

$$\mathbf{a(B}_k + \mathbf{C}_k) = \mathbf{aB}_k + \mathbf{aC}_k.$$

Separating parts of different grades, we get:

$$\mathbf{a} \cdot (\mathbf{B}_k + \mathbf{C}_k) = \mathbf{a} \cdot \mathbf{B}_k + \mathbf{a} \cdot \mathbf{C}_k \text{ and}$$

$$\mathbf{a} \wedge (\mathbf{B}_k + \mathbf{C}_k) = \mathbf{a} \wedge \mathbf{B}_k + \mathbf{a} \wedge \mathbf{C}_k.$$

This is a somewhat more generalized derivation of the distributive rule.

Separating a multivector into parts of different grade can be proven very useful.

Let us introduce a special notation for the separation of a multivector into its component blades. $\mathbf{<A>_r}$ is the r-vector part of the multivector $\mathbf{A}$.

If $\mathbf{A} = \mathbf{abc}$, then $\mathbf{<abc>_r} = \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ is the trivector part, and

$$\mathbf{<abc>_r} = \mathbf{a(b \cdot c) + a \cdot (b \wedge c)} \text{ is the vector part.}$$

Lastly, the sum of the r-vector parts is expressed by

$$A = \Sigma_r \mathbf{<A>_r} = \mathbf{<A>_0} + \mathbf{<A>_1} + \mathbf{<A>_2} + \mathbf{<A>_3} .$$

Geometric Algebra in 1-d

Every oriented line l has a directional vector $\mathbf{a}$. All vectors on l can be represented by multiplying $\mathbf{a}$ with a scalar α such that there is a vector $\mathbf{x} = \alpha\mathbf{a}$ which includes all vectors on l. This is the parametric equation of a line. The vectors $\mathbf{x}$ can be oriented either positively, $\mathbf{x} \cdot \mathbf{a} > 0$ or $\mathbf{x} \cdot \mathbf{a} < 0$. The unit vector $\mathbf{u} = \mathbf{a}\,|\mathbf{a}|^{-1}$.

Outer multiplication of $\mathbf{x} = \alpha\mathbf{a}$ by $\mathbf{a}$ such that $\mathbf{x} \wedge \mathbf{a} = \alpha(\mathbf{a} \wedge \mathbf{a})$ shows that $\mathbf{x} \wedge \mathbf{a} = 0$ and is the nonparametric equation of a line. The $\mathbf{a}$-line is the solution set$\{\mathbf{x}\}$of this equation.

If $\mathbf{xa} = \mathbf{x} \cdot \mathbf{a}$, and if $\alpha = \mathbf{x} \cdot \mathbf{a}^{-1}$ then multiplying each side by $\mathbf{a}^{-1}$, we get $\mathbf{x} = \alpha\mathbf{a}$.

Geometric Algebra in 2-d

If $\mathbf{B}$ is a bivector then the set of all vectors $\mathbf{x}$ which satisfy the equation

$$\mathbf{x} \wedge \mathbf{B} = 0$$

is a 2 dimensional vector space or plane. It is analogous to the equation of a line with its set $\mathbf{a}$ of all vectors on the line l. If $\boldsymbol{I}$ is the unit directional bivector of the plane $\mathbf{x} \wedge \mathbf{B} = 0$, then

$$\mathbf{B} = \mathrm{B}\boldsymbol{I}.$$

B can be divided out to form $\mathbf{x} \wedge \mathrm{B}\boldsymbol{I} = 0$ such that

$$\mathbf{x} \wedge \boldsymbol{I} = 0.$$

So, every non-zero scalar multiple of $\boldsymbol{I}$ also determines the plane of $\boldsymbol{I}$. The bivector $\boldsymbol{I}$ is known as the pseudoscalar of the plane and is the direction of the plane segment or unit directed area, a triangular area that has orientation.

The parametric equation for the $\boldsymbol{I}$-plane can be derived from $\mathbf{x} \wedge \mathbf{B} = 0$ by factoring $\boldsymbol{I}$ into the product

$$\boldsymbol{I} = \mathbf{i\,j} = \mathbf{i} \wedge \mathbf{j} = -\,\mathbf{j\,i},$$

where **i** and **j** are orthogonal unit vectors such that $\mathbf{i} \cdot \mathbf{j} = \mathbf{0}$, and $\mathbf{i} = \mathbf{j} = 1$. If $\mathbf{x}\,\boldsymbol{I} = \mathbf{x} \wedge \boldsymbol{I}$, then

$$\mathbf{x\,i\,j} = \mathbf{x(i \cdot j)} = \mathbf{x \cdot i\,j - x \cdot j\,i}.$$

Multiplying this on the right by **j i** we obtain

$$\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j},$$

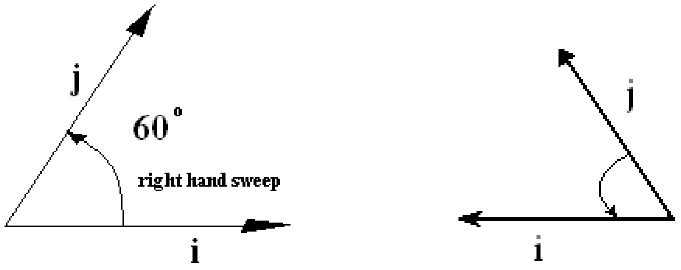
where $x_1 = \mathbf{x} \cdot \mathbf{i}$, and $x_2 = \mathbf{x} \cdot \mathbf{j}$. x_1 and x_2 are the triangular components of the vector **x** with respect to the basis {**i, j**} and with each new value of the x_1 and x_2 pair is a new vector **x**. Because $\mathbf{i} \cdot \mathbf{j} = 0$ then **i** and **j** are orthogonal.

Consider a plane which is spanned by 2 orthonormal vectors **i** and **j**. These basis vectors satisfy

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{1},\ \mathbf{i} \cdot \mathbf{j} = \mathbf{0}, \text{ and } \mathbf{i} = \mathbf{j}.$$

The next entity present in a 2 dimensional algebra is the bivector $\mathbf{i} \wedge \mathbf{j}$. By convention, bivectors are right-handed, so that **i** sweeps onto **j** in a right-handed sense when viewed from above. But if we keep the convention of **j** always being vertical, **j** sweeps into **i** either way.

the geometric product $\mathbf{i} \wedge \mathbf{j}$



classical arrangement also allowed

Let $\boldsymbol{I} = \mathbf{i} \wedge \mathbf{j} = \mathbf{i\,j}$, where $\boldsymbol{I}$ is called a pseudoscalar or unit bivector. The full algebra is spanned by

1	{**i, j**}	$\mathbf{i} \wedge \mathbf{j}$
1 scalar,	2 unit vectors,	and 1 bivector.

We denote this algebra by G_2. The law of multiplication for G_2 is that the geometric product

$$\mathbf{i\,j} = \mathbf{i \cdot j} + \mathbf{i} \wedge \mathbf{j} = \mathbf{i} \wedge \mathbf{j}\,.$$

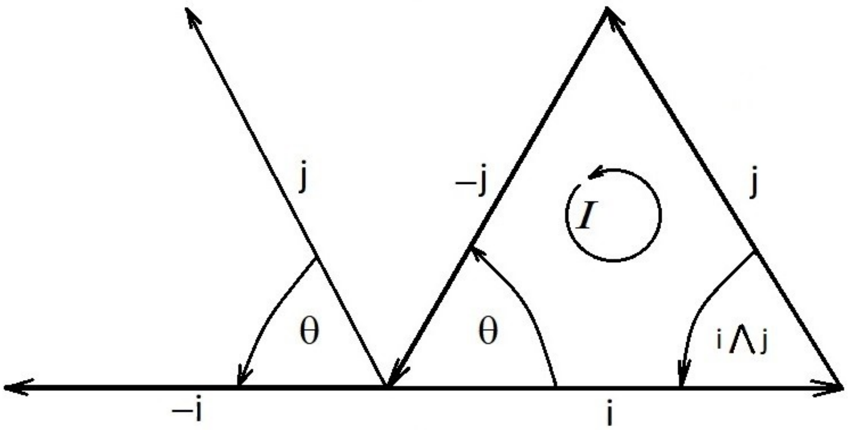
That is, for orthogonal vectors, the geometric product is a pure bivector. Because of the anticommutativity property of bivectors, $\mathbf{i\,j} = \mathbf{i} \wedge \mathbf{j} = \mathbf{j} \wedge - \mathbf{i} = -(\mathbf{j\,i})$. In Geometric Algebra, orthonormal vectors anticommute.

We can now form products from the right and from the left. Multiplying a vector by a bivector from the left, we reverse the vector. Let **i** be multiplied by **i j** on the left.

$$\mathbf{(i\,j)\,i = (\,j\,[-\,i]\,)\,i = j(-\,i\,i) = -\,j\,(i\,i) = -\,j},$$

$$\text{or } \mathbf{(i\,j)\,i = -\,(j\,i)\,i = -\,j\,(i\,i) = -\,j.}$$

i has been rotated counterclockwise 60^o into **– j**. Since $\boldsymbol{I} = \mathbf{i} \wedge \mathbf{j} = \mathbf{i\,j}$, we see that ***I*** is not only a unit directed area but is the generator of rotations. Notice that **j** => – **j** remains unchanged in that it remains vertical, but it is rotated 60^o. It only changes sides of the equilateral triangle. That is the convention.



Let **j** be rotated by **i j** on the left.

$$\mathbf{(i\,j)\,j = i(j\,j) = i}.$$

If **j** is rotated from the left, it is rotated 60^o, but changes into **i**. We see that **j** is rotated into **i** under a rotation by a bivector on the left.

Similarly, acting from the right,

$$\mathbf{i\,(i\,j) = (i\,i)\,j = j},$$

$$\mathbf{j\,(i\,j) = -\,(j\,j)\,i = -\,i},$$

multiplying from the right rotates a vector clockwise 60^o. In the above illustrations, replace i's with j's and visa versa.

Complex Numbers

The final product in the algebra is the triangle of the unit bivector.

$$\boldsymbol{I}^{\mathbf{2}} = (\,i \wedge j\,)^2 = i\,j\,i\,j = -\,i\,i\,j\,j = -\,1.$$

We have discovered a purely geometric quantity which triangles to – 1. Three successive left or right multiplications of a vector by **i j** rotates the vector through 180^o which is equivalent to multiplying by – 1. But only (i j) i or j (i j) will transform into $\boldsymbol{I}^{\mathbf{2}}$ or – 1.

Taking (i j) i = – j and

multiplying each side by j,

$$(i\,j)\,i\,j = -\,j\,j,$$

we have

$$\boldsymbol{I}^2 = -1.$$

Then taking

$$j\,(i\,j) = -\,i$$

and multiplying each side by i,

$$i\,j\,(i\,j) = -\,i\,i,$$

we again have

$$\boldsymbol{I}^2 = -1.$$

This cannot be done with (i j)j = i and i (i j) = j. All you wind up with is 1 = 1.

It appears that the bivector triangled brings the same result as an imaginary number triangled. That is, $\boldsymbol{I}^2 = -1$. As functions, they both rotate a vector 60°. The combination of a scalar and a bivector are naturally formed using the geometric product and can be viewed as a complex number

$$Z = u + v\,i\,j = u + \boldsymbol{I}\,v.$$

Every complex number has a real and an imaginary part.

In G_2, vectors are grade-1 objects,

$$x = ui + vj.$$

The mapping between this vector x and the complex number Z is simply premultiplying by i.

$$xi = uii + vij = u + \boldsymbol{I}\,v = Z.$$

Using this method, vectors can be interchanged with complex numbers.

Rotors

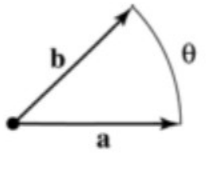
$\boldsymbol{I}$ is not only an operator, such that $\boldsymbol{I}\,\mathbf{a} = -\mathbf{a}\boldsymbol{I} = \mathbf{b}$, and since the solution of $\mathbf{ab} = \mathbf{C}$ is $\mathbf{b} = \mathbf{a}^{-1}\,\mathbf{C}$, the bivector $\boldsymbol{I}$ has the following equations and depictions for $\mathbf{a}^2 = \mathbf{b}^2 = \mathbf{1}$:

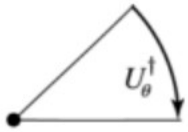
$$\boldsymbol{I} = \mathbf{ba}^{-1} \text{ (counterclockwise sense) and}$$

$$-\boldsymbol{I} = \mathbf{a}^{-1}\mathbf{b} \text{ (clockwise sense).}$$

$\boldsymbol{I} = \mathbf{ba}$ generalizes into the concept of a **rotor** U_θ, produced by the product ba of unit vectors with relative direction θ.

Rotor $U_\theta = \mathbf{ba}$ is depicted as a directed arc on the unit circle. Reverse $U_\theta' = \mathbf{ab}$.





Sin and Cos functions are defined from products of unit vectors.

$\mathbf{a}^2 = \mathbf{b}^2 = 1, \boldsymbol{I}^2 = -1$

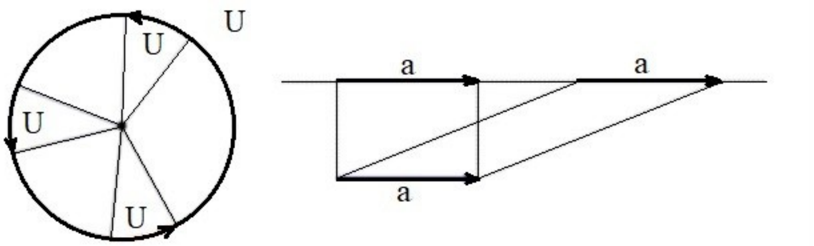
$\mathbf{b} \cdot \mathbf{a} = \cos\theta$

$\mathbf{b} \wedge \mathbf{a} = \boldsymbol{I} \sin\theta$

Rotor:

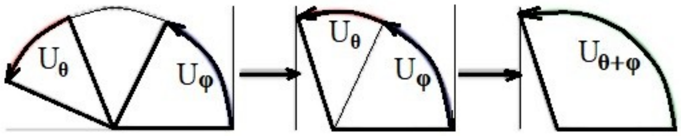
$U_\theta = \mathbf{ba} = \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \wedge \mathbf{a} = \cos\theta + \boldsymbol{I}\sin\theta = e^{i\theta}$ (note: when ***I*** is an exponent I use **i**)

Rotors being equivalent to directed arcs are like vectors being equivalent to directed line segments:

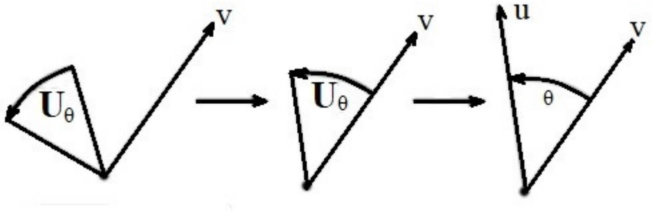


The **product of rotors** is equivalent to the addition of angles:

$U_\theta U_\varphi = U_{\theta+\varphi}$ or $e^{i\theta}e^{i\varphi} = e^{i(\theta+\varphi)}$



Rotor-Vector Product is a vector: $U_\theta \mathbf{v} = e^{i\theta}\mathbf{v} = \mathbf{u}$



Rotors as Complex Numbers:

$z = \lambda U = \lambda e^{i\theta} = \mathbf{ba}$

complex conjugate:

$z' = \lambda U' = \lambda e^{-i\theta} = \mathbf{ab}$

$zz' = (\mathbf{ba})(\mathbf{ab}) = \mathbf{a}^2\mathbf{b}^2 = \mathbf{z}^2$

Modulus: $\mathbf{z} = \lambda = |\mathbf{a}||\mathbf{b}|$

Complex Number Notation: (a special case of spinors in 3-d)

$z = z + \boldsymbol{I}z = \mathbf{ba}$

$z = \frac{1}{2}(z + z') = \mathbf{b} \cdot \mathbf{a}$

$\boldsymbol{I}z = \frac{1}{2}(z + z') = \mathbf{b} \wedge \mathbf{a}$

Vector Symptoms

"One barrier to developing the vector concept is the fact that the correspondence between vector and directed line segment has many different interpretations in modeling properties of real objects and their motions:

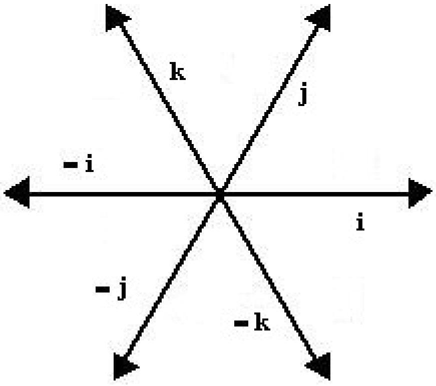
- *Abstract depiction* of vectors as manipulatable arrows has no physical interpretation, though it can be intuitively helpful in developing an abstract geometric interpretation.
- *Vectors as points* designate *places* in a Euclidean space or with respect to a physical reference frame. Requires designation of a distinguished point (the *origin*) by the zero vector.
- *Position vector* **x** for a particle which can "move" along a *particle trajectory* $\mathbf{x} = \mathbf{x}(t)$ must be distinguished from places which remain fixed.
- Kinematic vectors, such as *velocity* $\mathbf{v} = \mathbf{v}(t)$ and *acceleration* are "tied" to particle position $\mathbf{x}(t)$. Actually, they are vector fields defined along the whole trajectory.
- *Dynamic vectors* such as momentum and force representing particle interactions.
- *Rigid bodies*. It is often convenient to use a vector **a** as a 1-d geometric model for a rigid body like a rod or a ruler. It s magnitude $a = |\mathbf{a}|$ is then equal to the length of the body, and its direction $\hat{\mathbf{a}}$ represents the body' s orientation or, better, its *attitude* in space. The endpoints of **a** correspond to ends of the rigid body, as expressed in the following equation

$$\mathbf{x}(\alpha) = \mathbf{x_0} + \alpha\mathbf{a} \quad \{0 \leq \alpha \leq 1\}$$

for the position vectors of a continuous distribution of particles in the body. Note the crucial distinction between curves (and their parametric equations) that represent particle paths and curves that represent particle paths and curves that represent geometric features of physical bodies."

David Hestenes

Geometric Algebra of 3 Dimensions



We now add a third vector k to our 2-d set { i , j}. A plane is spanned by 3 orthonormal vectors i, j, k. All three vectors are assumed to be orthonormal, so they all anticommute. From these three vectors are generated three independent bivectors ij, jk, and ik. This is the expected number of independent planes in 3-D space.

The expanded algebra gives a number of new products to consider. One is the product of a bivector and an orthogonal vector,

$$(i \wedge j)k = ijk.$$

This corresponds to the bivector, a plane, $i \wedge j$, along the vector k. The result is a three dimensional volume element called a trivector $i \wedge j \wedge k$. The same result can be seen as $j \wedge k$ sweeps along i. Generalizing, $(a \wedge b)\, c = abc$. This gives us the 12 bivectors:

clockwise	counterclockwise
i(–j)k, –jki, ki(–j), –ij(–k), j(–k)(–i), –k(–i)j,	ik(–j), –jik, k(–j)i, –i(–k)j, j(–i)(–k), –kj(–i)

But some of these can be eliminated, as they are the same triangle, giving us 4 basic trivectors:

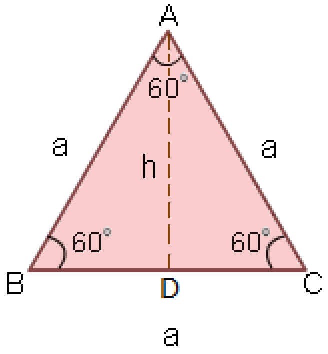
clockwise	counterclockwise
i(–j)k, –ij(–k),	ik(–j), –i(–k)j

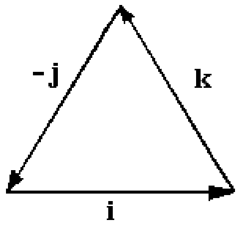
(Instead of ordering the vectors, only the signs have been ordered such that – + +, + – – ,– + –, + – +, + – –, – – +, + + +, and – – – , etc. There are actually 12 combinations, but 8 of them are not trivectors in that they do not exhibit circuitry.)

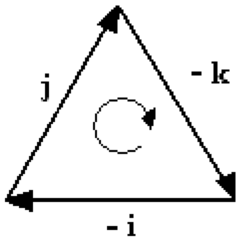
The basis vectors i, j, k satisfy

$i^2 = j^2 = k^2 = 1$, $i \cdot j = 0$, $j \cdot k = 0$, $k \cdot i = 0$, $\boldsymbol{I} = (i \wedge j \wedge k) = i\,j\,k$, $\boldsymbol{I}^2 = -1$ and $i - j = k$,

$-i + j = -k$, and etc., using all the above combinations,







which are called the properties of the algebra G_3, which is spaned by

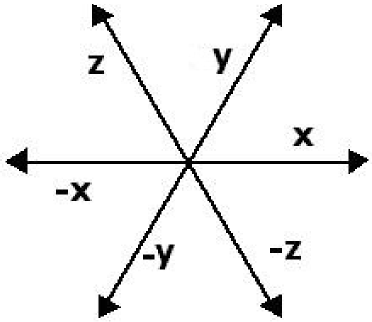
1	{i, j, k}	{i ∧ j, j ∧ k, k ∧ i}	{i ∧ j ∧ k}
1 scalar	3 unit vectors	3 bivectors	1 trivector

The other main property of G_3 is that it is antisymmetric on every pair of vectors,

$$a\wedge b\wedge c\wedge = -b\wedge a\wedge c = b\wedge c\wedge a = \text{etc.}$$

Swapping any two vectors reverses the orientation of the product.

The properties of the trivector ***I***, or as it is sometimes called, a pseudoscalar, or directed volume, is that it is right handed. In other words, $i - j = k$. Yet, in the 60° coordinate system, there is a reflection $-i + j = -k$, which, when combined with ***I*** produces a hexagonal coordinate system such that $|x_i| + |y_j| = |z_k|$.



Consider the product of a vector and pseudoscalar, i***I*** = i(ijk) = jk. This returns a bivector, the plane orthogonal to the original vector. Multiplying from the left, (ijk)i = jk, we find an independence of order. It follows then that I commutes with all the elements in the algebra. If a is any vector, then a ***I*** = ***I*** a. This is true with of the pseudoscalar in all odd dimensions. In even dimensions, the pseudoscalar anticommutes with all vectors as we saw in G_2.

Each of the basis bivectors can be expressed as the product of the pseudoscalar and what is known as the dual vector

$$ij = \boldsymbol{I}\,k, \quad jk = \boldsymbol{I}\,i, \quad ki = \boldsymbol{I}\,j$$

This operation is known as duality transformation. Also,

$$a\,\boldsymbol{I} = \boldsymbol{a} \cdot \boldsymbol{I}$$

can be understood as a projection onto the component of ***I*** orthogonal to a.

The product of a bivector and a pseudoscalar

$$\boldsymbol{I}(i \wedge j) = \boldsymbol{I}\,ijkk = \boldsymbol{I}\,\boldsymbol{I}\,k \ = -\,k \qquad \text{(note } kk = 1\text{)}$$

the bivector being mapped onto a vector via the duality operation.

The square of the pseudoscalar is $\boldsymbol{I}^2 =$ ijkijk = ijij = – iijj = – 1 .

Chapter Eight

Numerology

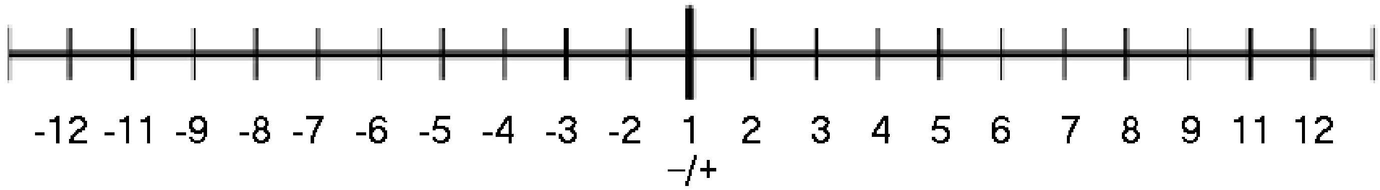
Using numbers to the base 9 is more logical

Most people, when they think of numbers to the base n include zero as the first digit. I am not going to include zero, and I am going to count by nines.

0		1	2	3	4	5	6	7	8	9		10
10		**11**	12	13	14	15	16	17	18	19		20
20		21	**22**	23	24	25	26	27	28	29		30
30		31	32	**33**	34	35	36	37	38	39		40
40		41	42	43	**44**	45	46	47	48	49		50
50		51	52	53	54	**55**	56	57	58	59		60
60		61	62	63	64	65	**66**	67	68	69		70
70		71	72	73	74	75	76	**77**	78	79		80
80		81	82	83	84	85	86	87	**88**	89		90
90		91	92	93	94	95	96	97	98	**99**		100 ...
100		111	112	113	114	115	116	117	118	119	…	**999**

Here is a table of numbers from 1 to 119 with the zero and 10’s stripped away. What we have is counting by nines. 99 is understandable, but what is 111? That’s 100 in base 10. What comes after 9, but 11? It is 10 in base 10.

I propose a number line with one as the origin, negative numbers to the left and positive numbers to the right.



All the 10’s are missing, viz., 10, 20, 30, 40, …, 100, 1000, 10,000, etc. But if you wanted to count by 10’s, you would count 11, 22, 33, 44, 55, 66, … etc. That isn’t strange when you realized that in base 10: 10/9 = 1.1111 … , 20/9 = 2.2222 … , 60/9 = 6.6666 … , etc. But in base 9: 11/9 = 1*9 +1 = 11, 22/9 = 2*9 +2 = 22, 33/9 = 3*9 +3 = 33, 66/9 = 6*9+6 = 66, etc. all whole numbers. (1*9 means one 9, 2*9 means two 9’s, etc.)

To me, that is logical or more reasonable.

The 9’s multiplication table is

1*9 = 9, 2*9 = 19, 3*9 = 29, … 6*9 = 59, etc. … 9^2 = 89, and 11*9 = 99, since 11 comes after 9 in counting by 9’s. (example: 2*9 means two nines, and 19 means two nines) This makes it a lot easier to count.

Each double number can be expressed by 9 * n + n. Every triple number, as 99 * n + n. Every quadruple, as 999 * n + n. So any number with a repeating digit is 9^i * j + j.

If you have a remainder r from a division, the first r behind the decimal would be r/9. The next remainder would be r/99, etc. If you want to represent how many 9s behind the decimal point, an example might be:

1/9 =.1, 1/9 + 1/99 =.11, 1/9 + 1/99 + 1/999 = .111, etc.

In other words, wherever there would be 10, 100, 1000, etc, you would have 9, 99, 999, etc.

Let N(– , +, *, /) be an algebra such that N includes subtraction, addition, multiplication, and division. An example for subtraction would be 1 – 1 = 11 – 1 = 9 or 91 – 1 = 9 and the carry is dismissed. Other examples are 2 – 2 = 12 – 2 = 9 or 92 – 2 = 9 and the carry is dismissed. This is true for all the digits. So 9 is the place holder instead of 0.

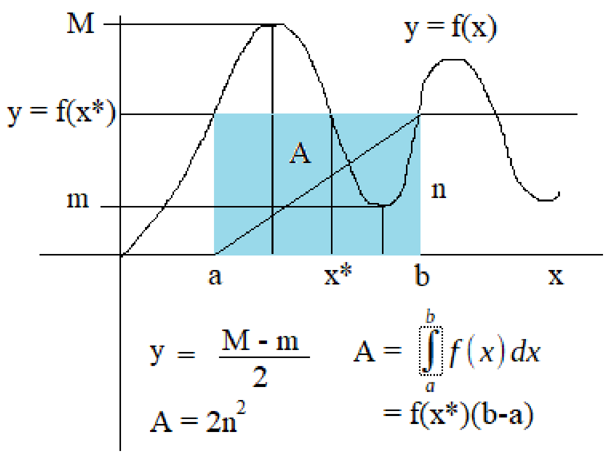
Here is another thing that is most logical. Remember that when we have things like 14 – 9 = 5, 15 – 9 = 6, or 17 – 9 = 8? Well, in base 9 numbers, 11 – 9 = 1, 12 – 9 = 2, 13 – 9 = 3, etc. There is a correlation between the first list of digits and the second list of digits, viz. in base 10, the lists miss the correlation by one digit. I call base 9 counting cleaner.

There is only one mystery, and that is subtraction. It doesn't make sense. In place of zero, we have – 1. Any number n – n = – 1. But if you want to know the distance between each digit and its negative counterpart, you can see below that the even numbers cycle with the odd numbers. There is a list of 5 even numbers alternating with 4 even numbers.

1 – 1 = – 1,
2 – 2 = – 3,
3 – 3 = – 5,
4 – 4 = – 7,
5 – 5 = – 9,
6 – 6 = – 12,
7 – 7 = – 14,
8 – 8 = – 16,
9 – 9 = – 18,
11 – 11 = – 21,
12 – 12 = – 23,
13 – 13 = – 25,
… etc.

What does subtracting columns look like? Using some random numbers,

$$\begin{array}{r} 384659 \\ -243647 \\ \hline 139912 \end{array} \qquad \begin{array}{r} 582914 \\ -243647 \\ \hline 337256 \end{array}$$



Appendix or Addendum

Applications

Chord vs Arc

The trigonometric functions in the hexagonal coordinate system is based upon the chords or the sides of the hexagon and not the inscribing circle around it. The constraints of the system are set up in this program or Python module called *HexMath.py*:

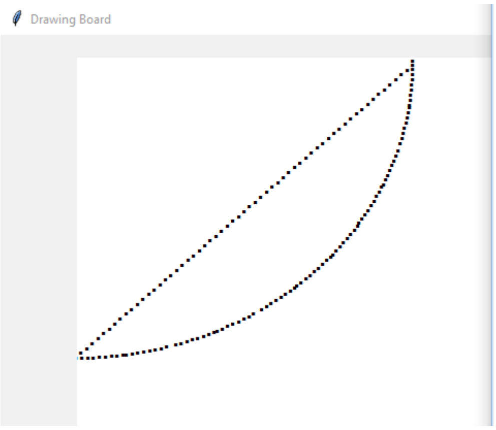
```
#!Python
"""
   Math functions to be used by other modules
   using hexagonal coordinate systems
   Estel Murdock
   23 October 2020
"""

def sin(n):
   x = 0
   if n > 0 and n < 60:
      x = n/60 #maps numbers 60 and below to numbers 0 to 60.
   elif n > 60:
      x = (n/60)%60  #maps numbers 60 and above to numbers between 60 to 0.
   elif n == 60:
      x = 1
   else:
      x = 0
   return x
def cos(n):
   x = 0
   if n > 0 and n < 60:
      x = (60 - n)/60 #maps numbers 60 and below to numbers between 0 to 60.
   elif n > 60:
      x = ((60 - n)/60)%60 #maps numbers 60 and above to numbers between 60 to 0.
   elif n == 60:
      x = 0
   else:
      x = 1
   return x
```

This module is used in another module called *Line1DDemo.py*:

```
#!Python3

"""
This is a drawing board for a rotated or rotating line
based upon a Hexagonal Coordinate system
Estel Murdkock
19 March 2021
"""
# import modules to use their functions
from tkinter import*
import HexMath as h
import time
import math as m

# canvas dimensions
canvas_width = 652
canvas_height = 568
bgcolor = 'white'

#create a drawing board
root = Tk()
root.wm_title('Drawing Board')
root.geometry("800x600")
canvas = Canvas(root, width=canvas_width, height=canvas_height, bg=bgcolor)
canvas.pack(pady=20)

# draw the points
for t in range(360):
    root.update()
    time.sleep(.11)

    #Hexagonal coordinates
    x = (canvas_width/2) * h.sin(t)
    #print('x = ', x)
    y = (canvas_height/2) * h.cos(t)

    #Cartisian Coordinates
    x1 = (canvas_width/2) * m.sin(t)
    #print('x = ', x)
    y1 = (canvas_height/2) * m.cos(t)
    #print('y = ', y)

    #draws a chord across an arc - coordinates are based upon the chord or side of a hexagon
```

```
    dot2 = canvas.create_text(x, y, font=("Times Roman", 24), text=".")

#draws an arc across a chord - coordinates are based upon radians along an arc
    dot2 = canvas.create_text(x1, y1, font=("Times Roman", 24), text=".")

root.mainloop()
```

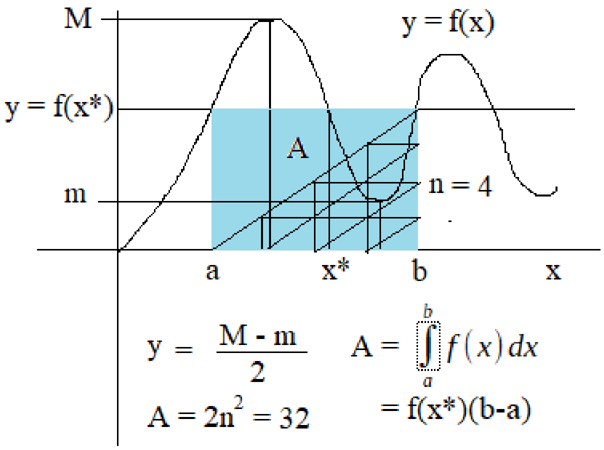
The output of this demo module shows the difference between the Cartesian, 90° style, coordinate system in which the trigonometric functions draw an arc and the Hexagonal, 60° style, coordinate system in which the trigonometric functions draw a chord of the arc or a side of a hexagon. The same values were given to the trig functions each time.

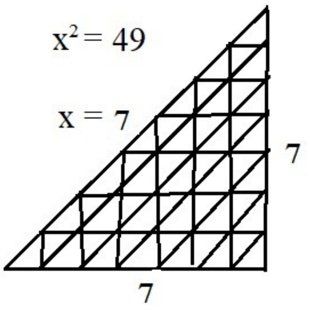


This is a Python Tkinter window, and as on any monitor screen, the point (0,0) is in the upper left hand corner with the positive x-axis extending across to the right and the positive y-axis extending down on the left.

Mean Value Theorem of Calculus

Integral calculus gives you the area under the curve on a graph, right? But this theorem fashions this area into a rectangle xy such that on the x axis, x = (b – a) and y = f(x*) which is the mean value between a high value M and low value m on the y axis, and the area is written as A = (b – a)f(x*) or xy. Now if you cut that rectangle diagonally into two triangles, and divide one of those triangles into similar triangles, each side of the original or larger triangle will be n units long. That is just like a square being divided into smaller squares or an equilateral triangle being divided into smaller similar triangles. That shows a one to one correspondence between the equilateral triangle and the square. Now since the length of each side is n, the area of the triangle is n^2, just like a square. So, the area of the rectangle, and thus the area under the curve, and thus, the definite integral between x = a and x = b on the x axis, is $2n^2$, since there are two triangles in the rectangle. This is a very general interpretation of integral calculus. Depending upon the scale of the x and y axis, and thus on the triangle sides, it could be any number as long as it is not infinite. Remember, you are counting triangles. So the area under the curve or the definite integral is $2n^2$ triangles.

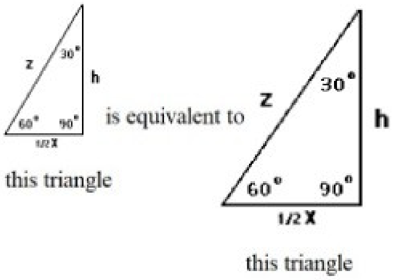




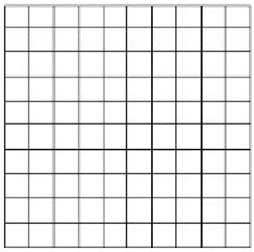
Each side of the original triangle will be n units long.

I explained that dividing a triangle into similar triangles is like dividing a square into smaller squares, so each triangle within the larger triangle is an exact copy of the larger triangle, only smaller, and each smaller triangle is the same size as all the other smaller triangles, and has the same angels in the same places as the larger triangle.

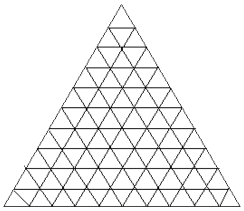
Similar triangles



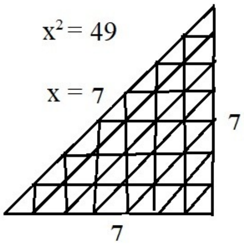
a square divided into similar squares



$x^2 = 10^2 = 100$

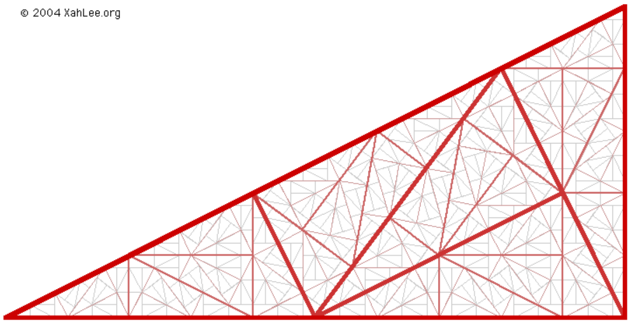


$x^2 = 10^2 = 100$
similar triangles



another triangle divided
into similar triangles

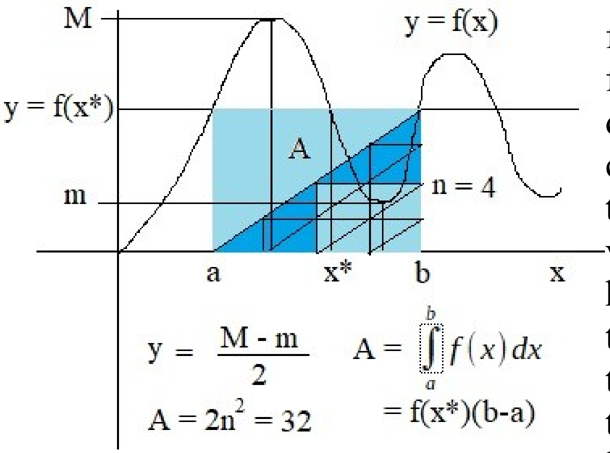
That term "similar triangles" comes from Euclidean geometry. For a specific example, suppose the rectangle under the curve f(x*) has dimensions 3x4. Divide along the long diagonal to get two 3-4-5 right triangles. Sure, you can cut the 3-4-5 triangle into two smaller triangles similar to the original 3-4-5 triangle. But the two smaller triangles cannot be congruent to each other (the proof is easy: cutting the 3-4-5 triangle in two parts requires picking a side and drawing perpendicular to that side to the opposite vertex in order to form the common edge of the two smaller triangles).



This is **not** what I mean.

Now if you cut that triangle in this manner, of course, you cannot obtain the area by counting similar triangles.

Take the 3x4 rectangle. For the mean value theorem to make sense, its height is f(x*) and its base is (b-a). You may say, that evidently, these are not the same value in general. But you are missing the point. There is no 3-4-5 triangle. There is only a 3-3-3, 4-4-4, or 5-5-5, or etc. triangle when you divide any triangle into similar triangles. I am talking about a paradigm shift here. You cannot measure in the old way, having each side measured separately. Each triangle is stretchable into an equilateral triangle, and an equilateral triangle is “equivalent" to a square in the way the area is found. A unit equilateral triangle and a unit square have the same area, no matter what the size. A unit triangle or square can be of any size. You choose the scale. Look at the right triangle above which has an area of 49. Its area is a pure number without reference to a ruler.



You must remember that I am generalizing f(x*)(b-a) to where f(x*) = (b-a). You assume that f(x*)(b-a) has already been found (that's the key) and divided in half diagonally. You find out what one half can be construed as and then double it to come back to the original value. My hypothesis is that the two values should be the same if you find the right scale. I have divided one of the triangles in two, giving 4 inner triangles, and then divided it into 4, giving 16 inner triangles. It can keep going up and up. By the way, a triangle cannot be divided into two similar triangles, but only 4. Also, Buckminister Fuller taught that every value can be expressed in whole numbers or very close to it. That's what I'm doing here. So twice 16 is 32, the area under the square. Obviously, this is the wrong scale, but just to make a point.

Inverse Trigonometric Functions and Their Derivatives

An inverse function g(y) = x is generally a reflection of another function f(x) = y across the line such as x = y (or across the axis of the plane) if and only if g(f(x)) = x.

That is, as $f: X \rightarrow Y$, then $g: Y \rightarrow X$. There must be a property that for every y in Y there must be one, and only one x in X so that $f(x) = y$ and for every x in X there must be one, and only one y in Y so that $g(y) = x$. This property ensures that a function will exist having the necessary relationship with f. This is called a one to one property.

If f has an inverse, we will denote it by f^{-1}.

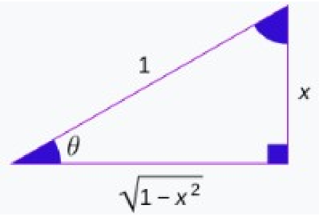
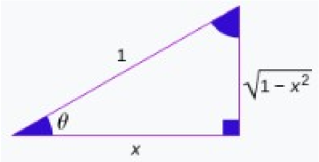
Not all functions have inverse functions. If you can draw a horizontal line through the curve of a function and come up with one or more values then there is a one to many property or relationship. It is not a reflection of the function.

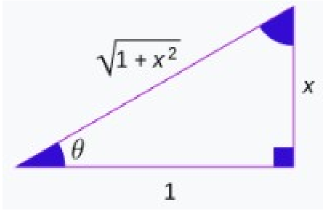
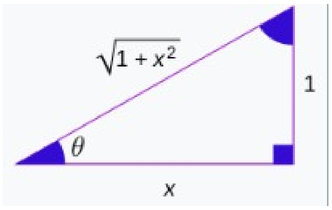
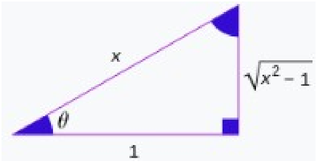
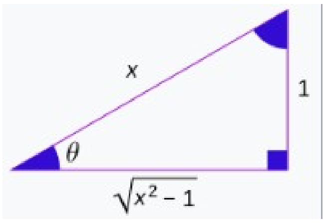
However, if we choose a function such as the sin function, we can have an inverse sin function over the interval of $[-\pi/2, \pi/2]$. This can be accomplished for a number of trigonometric functions.

Range of usual principal value	Domain of x for real result
$-\pi/2 \le \sin^{-1}(x) \le \pi/2$	$-1 \le x \le 1$
$0 \le \cos^{-1}(x) \le \pi$	$-1 \le x \le 1$
$-\pi/2 < \tan^{-1}(x) < \pi/2$	all real numbers
$0 < \cot^{-1}(x) < \pi$	all real numbers
$0 \le \sec^{-1}(x) \le \pi$	$x \le -1$ or $1 \le x$
$-\pi/2 \le \csc^{-1}(x) \le \pi/2$	$x \le -1$ or $1 \le x$

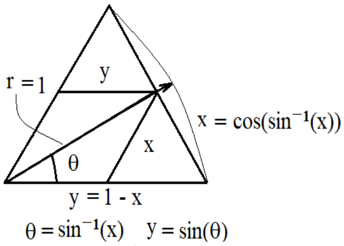
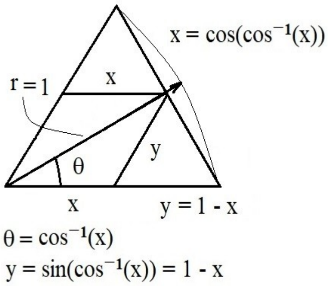
I want to show the derivatives of these functions and to explain the complications that arise from using the 90° coordinate system and applying the Pythagorean Theorem.

Relationships between trigonometric functions and inverse trigonometric functions

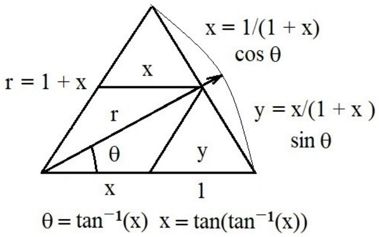
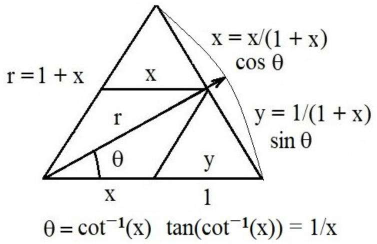
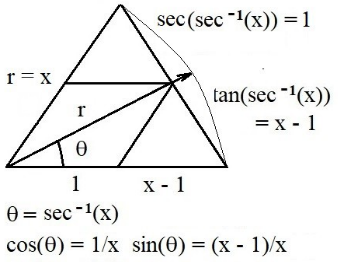
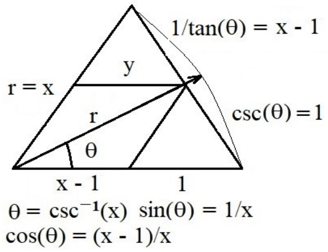
θ	**sin(θ)**	**cos(θ)**	**tan(θ)**	**diagrams**
$\sin^{-1}(x)$	$\sin(\sin^{-1}(x)) = x$	$\cos(\sin^{-1}(x))$ $= (1 - x^2)^{1/2}$	$\tan(\sin^{-1}(x)) =$ $x/(1 - x^2)^{1/2}$	 $\theta = \sin^{-1}(x)$
$\cos^{-1}(x)$	$\sin(\cos^{-1}(x)) =$ $(1 - x^2)^{1/2}$	$\cos(\cos^{-1}(x)) =$ x	$\tan(\cos^{-1}(x)) =$ $(1 - x^2)^{1/2}/x$	 $\theta = \cos^{-1}(x)$

$\tan^{-1}(x)$	$\sin(\tan^{-1}(x)) =$ $x/(1 + x^2)^{1/2}$	$\cos(\tan^{-1}(x)) =$ $1/(1 + x^2)^{1/2}$	$\tan(\tan^{-1}(x)) = x$	 $\theta = \tan^{-1}(x)$
$\cot^{-1}(x)$	$\sin(\cot^{-1}(x)) =$ $1/(1 + x^2)^{1/2}$	$\cos(\cot^{-1}(x)) =$ $x/(1 + x^2)^{1/2}$	$\tan(\cot^{-1}(x)) =$ $1/x$	 $\theta = \cot^{-1}(x)$
$\sec^{-1}(x)$	$\sin(\sec^{-1}(x))$ $= (x^2 – 1)^{1/2}/x$	$\cos(\sec^{-1}(x)) =$ $1/x$	$\tan(\sec^{-1}(x)) =$ $(x^2 – 1)^{1/2}$	 $\theta = \sec^{-1}(x)$
$\csc^{-1}(x)$	$\sin(\csc^{-1}(x)) =$ $1/x$	$\cos(\csc^{-1}(x)) =$ $(x^2 – 1)^{1/2}/x$	$\tan(\csc^{-1}(x)) =$ $1/(x^2 – 1)^{1/2}$	 $\theta = \csc^{-1}(x)$

Let us now change the geometry from 90° coordinates to 60° coordinates. The right triangles above are replaced by equilateral equiangular triangles.

$\sin^{-1}(x)$	$\sin(\sin^{-1}(x)) = x$	$\cos(\sin^{-1}(x)) =$ $1 – x$	$\tan(\sin^{-1}(x)) =$ $x/(1 – x)$	 $\theta = \sin^{-1}(x)$
$\cos^{-1}(x)$	$\sin(\cos^{-1}(x)) =$ $1 – x$	$\cos(\cos^{-1}(x)) =$ x	$\tan(\cos^{-1}(x)) =$ $(1 – x)/x$	

$\theta = \cos^{-1}(x)$

$\tan^{-1}(x)$	$\sin(\tan^{-1}(x)) = x/(1 + x)$	$\cos(\tan^{-1}(x)) = 1/(1 + x)$	$\tan(\tan^{-1}(x)) = x$	 $\theta = \tan^{-1}(x)$
$\cot^{-1}(x)$	$\sin(\cot^{-1}(x)) = 1/(1 + x)$	$\cos(\cot^{-1}(x)) = x/(1 + x)$	$\tan(\cot^{-1}(x)) = 1/x$	 $\theta = \cot^{-1}(x)$
$\sec^{-1}(x)$	$\sin(\sec^{-1}(x)) = (x - 1)/x$	$\cos(\sec^{-1}(x)) = 1/x$	$\tan(\sec^{-1}(x)) = (x - 1)$	 $\theta = \sec^{-1}(x)$
$\csc^{-1}(x)$	$\sin(\csc^{-1}(x)) = 1/x$	$\cos(\csc^{-1}(x)) = (x - 1)/x$	$\tan(\csc^{-1}(x)) = 1/(x - 1)$	 $\theta = \csc^{-1}(x)$

Using the 60° coordinate system, the Pythagorean Theorem is taken out of the picture. From the equilateral triangle, $r = x + y$ takes the place of $r^2 = x^2 + y^2$. That also means that $\sin\theta + \cos\theta = 1$ where $60^o > \theta > 0^o$ replaces $\sin^2\theta + \cos^2\theta = 1$, where $90^o > \theta > 0^o$.

Simplifying the function $\cos(\sin^{-1}(x))$. We start with the identity

$$\sin\theta + \cos\theta = 1.$$

Change this to

$$\cos\theta = 1 - \sin\theta$$

and substitute $\theta = \sin^{-1}(x)$ to obtained

$$\cos(\sin^{-1}(x)) = 1 - \sin(\sin^{-1}(x)) = 1 - x,$$

so $$\cos(\sin^{-1}(x)) = 1 - x.$$

Likewise for

$$\sec\theta - \tan\theta = 1.$$

Change this to

$$\sec\theta = 1 + \tan\theta$$

and substitute $\theta = \tan^{-1}(x)$ to obtained

$$\sec(\tan^{-1}(x)) = 1 + \tan(\tan^{-1}(x)) = 1 + x,$$

so $$\sec(\tan^{-1}(x)) = 1 + x.$$

The other trigonometric functions of inverse trigonometric functions are likewise found from the right identities which do not require the Pythagorean Theorem.

What are the derivatives of the inverse trigonometric functions?

For a 90º Coordinate System	For a 60º Coordinate System
$d[\sin^{-1}(x)]/dx = 1/(1 - x^2)^{1/2}$	$d[\sin^{-1}(x)]/dx = 1/(1 - x)$
$d[\cos^{-1}(x)]/dx = - 1/(1 - x^2)^{1/2}$	$d[\cos^{-1}(x)]/dx = - 1/(1 - x)$
$d[\tan^{-1}(x)]/dx = 1/(1 + x^2)$	$d[\tan^{-1}(x)]/dx = 1/(1 + x)$
$d[\cot^{-1}(x)]/dx = - 1/(1 + x^2)$	$d[\cot^{-1}(x)]/dx = - 1/(1 + x)$
$d[\sec^{-1}(x)]/dx = 1/x(x^2 - 1)^{1/2}$	$d[\sec^{-1}(x)]/dx = 1/x(x - 1)$
$d[\csc^{-1}(x)]/dx = - 1/x(x^2 - 1)^{1/2}$	$d[\csc^{-1}(x)]/dx = - 1/x(x - 1)$

Proof:

Let $y = \sin^{-1}(x)$	Let $y = \sin^{-1}(x)$
so that $x = \sin y$ where $-\pi/2 < y < \pi/2$	so that $x = \sin y$ where $-1 < y < 1$
$d/dx[\sin^{-1}(x)] = dy/dx = 1/(dx/dy) = 1/\cos y$	$d/dx[\sin^{-1}(x)] = dy/dx = 1/(dx/dy) = 1/\cos y$
$= 1/\cos(\sin^{-1}(x)) = 1/(1 - x^2)^{1/2}$	$= 1/\cos(\sin^{-1}(x)) = 1/(1 - x)$
Let $y = \tan^{-1}(x)$	Let $y = \tan^{-1}(x)$
so that $x = \tan y$ where $-\pi/2 < y < \pi/2$	so that $x = \tan y$ where $-1 < y < 1$
$d/dx[\tan^{-1}(x)] = dy/dx = 1/(dx/dy) = 1/\sec^2 y$	$d/dx[\tan^{-1}(x)] = dy/dx = 1/(dx/dy) = 1/\sec^2 y$
$= 1/\sec^2(\tan^{-1}(x)) = 1/(1 + x^2)$	$= 1/\sec^2(\tan^{-1}(x)) = 1/(1 + x)$

Let $y = \sec^{-1}(x)$

so that $x = \sec y$ where $0 \le y < \pi/2$ or

$\pi \le y < 3\pi/2$

$d/dx[\sec^{-1}(x)] = dy/dx = 1/(dx/dy)$

$= 1/(\sec y \tan y)$

$= 1/(\sec(\sec^{-1}(x))\tan(\sec^{-1}(x))) = 1/(x(x^2 - 1)^{1/2})$

Let $y = \sec^{-1}(x)$

so that $x = \sec y$ where $0 \le y < 1$ or

$3 \le y < (3 + 3/2)$

$d/dx[\sec^{-1}(x)] = dy/dx = 1/(dx/dy)$

$= 1/(\sec y \tan y)$

$= 1/(\sec(\sec^{-1}(x))\tan(\sec^{-1}(x))) = 1/x(x - 1)$

The remaining derivatives can be obtained from these three derivatives by using appropriate identities.

Conversion from degrees to radians is usually $30^\circ = \pi/6$, but for hexagonal $30^\circ = 3/6 = 1/2$

$45^\circ = \pi/4$	$45^\circ = 3/4$
$60^\circ = \pi/3$	$60^\circ = 3/3 = 1$
$90^\circ = \pi/2$	$90^\circ = 3/2$

Through these comparisons between the 90º coordinate system and the 60º or hexagonal way of doing things, changing the geometry from right triangles to equilateral triangles, it is seen that the figures are simpler in that squares and square roots are not used unless you are talking about areas. But if we can prove that derivatives and integrals can be replaced by powers and roots, that would make things a lot easier, and mathematics becomes simple.

Now if you take

$$d[\sin^{-1}(x)]/dx = 1/(1 - x),$$

multiply both sides by dx,

$$d[\sin^{-1}(x)] = dx/(1 - x),$$

and then integrate both sides, we have

$$\int dx/(1 - x) = \sin^{-1}(x) + C.$$

In a similar manner, all the other inverse trigonometric functions can be listed as the integrals of their derivatives.

$$-\int dx/(1 - x) = \cos^{-1}(x) + C.$$

$$\int dx/(1 + x) = \tan^{-1}(x) + C.$$

$$-\int dx/(1 + x) = \cot^{-1}(x) + C.$$

$$\int dx/x(x - 1) = \sec^{-1}(x) + C.$$

$$-\int dx/x(x-1) = \csc^{-1}(x) + C.$$

You can see that the $\cos^{-1}$, $\cot^{-1}$, and $\csc^{-1}$ integrals are just the negative of the $\sin^{-1}$, $\tan^{-1}$, and $\sec^{-1}$ integrals.

The Taylor Series

The Taylor series $\sum_{x=0}^{\infty} x^2 = 1/(1 + x)$ is reminiscent of the inverse trigonometric functions :

$$\cos(\tan^{-1}(x)) = 1/(1 + x),$$

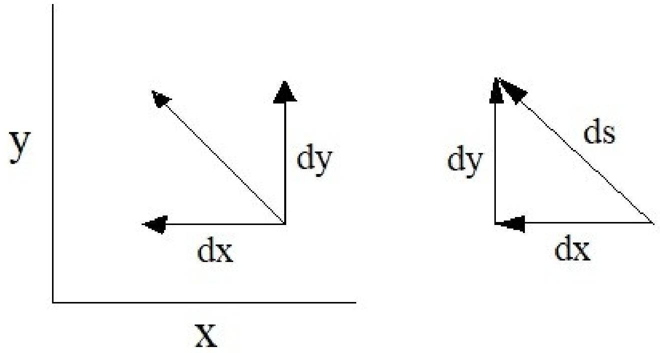
$$\sin(\cot^{-1}(x)) = 1/(1 + x), \text{ and}$$

$$\tan(\csc^{-1}(x)) = 1/(x - 1).$$

The Addition of Vectors and the Death of the Pythagorean Theorem

Here is another reason why, if you change geometry, you don't need the Pythagorean Theorem.

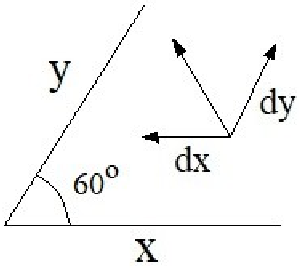
Consider a translation of a point in the 90º coordinate system such as a distance traveled.

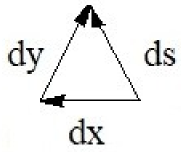


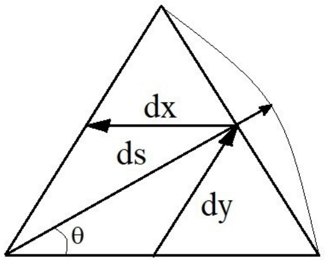
The magnitude ds of the vector **s** for the distance is usually found by using the Pythagorean theorem,

$$d^2s = d^2x + d^2y.$$

But if we change the geometry, using the 60º coordinate system, the solution becomes simple.







bivector or regular vectors

We have instead,

$$ds = dx + dy.$$

This is because of the universality of the coordinates in a hexagonal grid instead of a square grid. In other words, ds is equal in length to a side of an equilateral triangle, but the general case is where the side ds, which can be at any angle, is cut into two sections dx and dy.

This is similar to the adding of vectors such as

$$d\mathbf{s} = d\mathbf{x} + d\mathbf{y}.$$

Note: A change in the height over a vector field is given by

$$d\varphi_{\mathbf{s}} = d\varphi_{\mathbf{x}} + d\varphi_{\mathbf{y}}$$

where each $d\varphi_{\mathbf{k}} = (d\varphi/dk)\ dk$ so that

$$d\varphi_{\mathbf{s}} = (d\varphi/dx)\ dx + (d\varphi/dx)\ dx,$$

or more exactly, using partial derivatives,

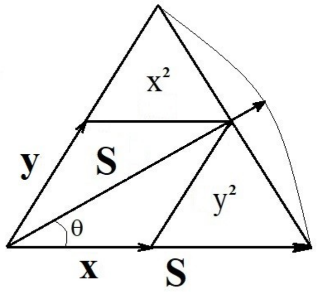
$$d\varphi_{\mathbf{s}} = (\delta\varphi/\delta x)\ dx + (\delta\varphi/\delta x)\ dx.$$

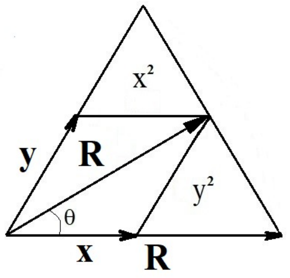
But to me, this is too complicated, so I have included the chapter on geometric algebra.

The Difference Between vectors S and R

Note: whereas **v** is a vector, let v denote a line segment.

There are two vectors **S** and **R** in describing the space within a hexagonal system.





The area of a triangle is either S^2 or R^2.

In one case, $S = \surd(S^2) = x + y$, and

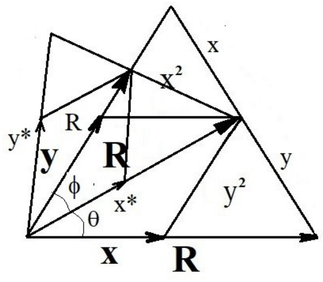
in the other case, $R = \surd(x^2 + xy + y^2)$.

Whereas $S^2 = f(x, y) = (x + y)^2 = x^2 + 2xy + y^2$,

$R^2 = f(x, y, \theta) = (x \cos \theta)^2 + x^*y^* + (y \sin \theta)^2$.

So when $S = x + y$,

$R = x \cos \theta + y \sin \theta$.



R starts out as the base of an equilateral triangle. As it rotates with an angle θ, R becomes the base of a new triangle which is at an angle of θ from the original triangle. R in the new triangle equals the sum of x^* and y^* such that $R = x^* + y^*$. But in relation to x and y in the old triangle, $x^* = x \cos \theta$ and $y^* = y \sin \theta$ so that the new

$R = x \cos \theta + y \sin \theta$. (see the paragraph below *The Length of r*)

It must be remembered, and this is a description of the triangles in the transcribed triangle, that $\cos \theta + \sin \theta = 1$, or $\cos \theta = 1 - \sin \theta$ and $\sin \theta = 1 - \cos \theta$ such that when one function approaches zero, the other is approaching one. As the x triangle decreases, the y triangle increases and visa versa. So R starts at a maximum, reaches a minimum and increases again to a maximum. The transcribed triangle where $R = x^* + y^*$ follows the same course. Conclusion: R can always be thought of as $R = x + y$, just as S is. Another conclusion is that whereas S^2 is static, R^2 is continually changing in an interval between two maximums a and b with a minimum c in the middle.

Proof of the Area of the Equilateral Triangle

Statement: In an equilateral triangle, the height will cut across the middle of the triangle, forming two right triangles and dividing the base in 2 equal halves.

Proof:

Step 1: Since all the 3 sides of the triangle are same,

$AB = BC = CA = a$

Step 2: Find the altitude of the △ABC.

Draw a perpendicular from point A to base BC, $AD \perp BC$ by using Pythagoras theorem in △ ADC.

$h^2 = AC^2 – DC^2 = a^2 - (a/2)^2$ [Because, $DC = a/2$] $= a^2 – a^2/4$

$h = \sqrt{}(a^2 – a^2/4)$

$h = (a\sqrt{}3)/2$

Step 3: We know that, Area of a triangle = 1/2 * Base * Height

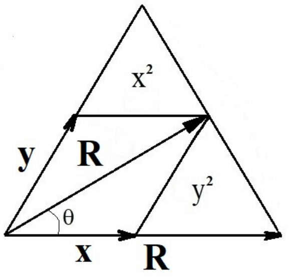
$= 1/2 * a * (a\sqrt{}3)/2 = (a^2\sqrt{}3)/4$

The area of an equilateral triangle = $(a^2\sqrt{}3)/4$.*

But, line 2 under *Step 2* is the fatal mistake. Without the Pythagorean Theorem, all you have is a^2, which is what you have in the hexagonal 60° coordinate system. This a^2 is the side a triangled, comes from triangling the side a of the triangle ADC, and counting similar triangles using triangular numbers, instead of counting squares which do not fit inside the triangle. You have to use the irrational√3 in order to make these squares fit. You get a smaller number when using the Pythagorean Theorem, but then you are not counting to the scale that fits the triangle. The multiplier that translates a 60° to a 90° result then would be (√3)/4.

*https://math.tutorvista.com/geometry/area-of-equilateral-triangle.html

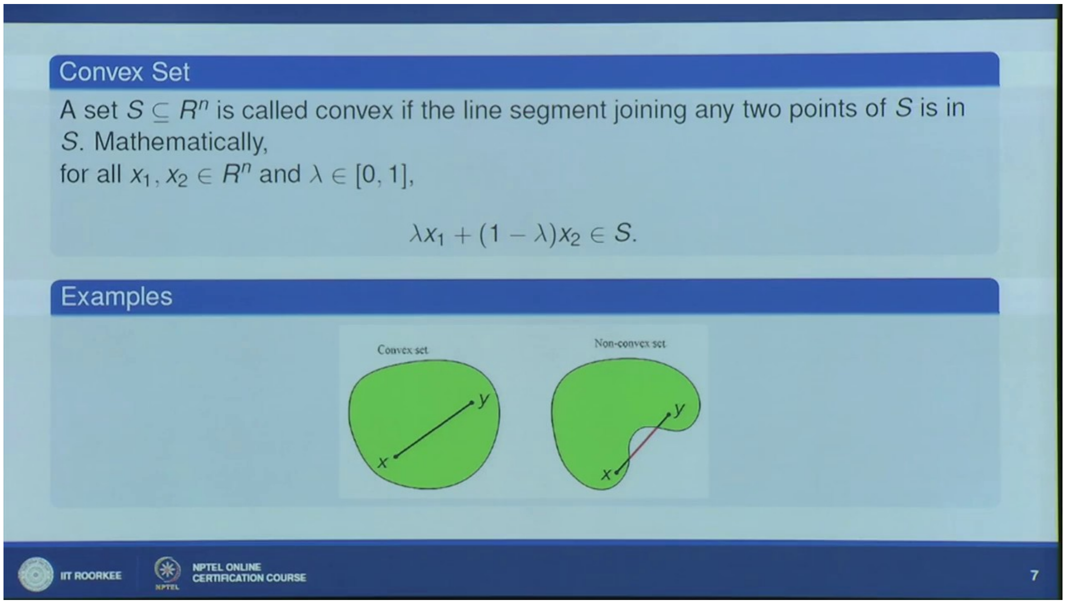
The Murdock Injunction



I will bring all of mathematics to its knees. The Murdock Injunction is that all triangles are equilateral triangles. They obey all the laws of equilateral triangles. These are this, that the side of the triangle is divided into two line segments by another line or vector coming from the opposite corner. This division creates two internal equiangular, equilateral triangles, the x triangle on top and the y triangle on the right side of the parallelogram encasing the said line or vector. The larger triangle is divided into a set of unit similar triangles, each having the same size and similar shapes and angles as the enclosing outer triangle. The number of these triangles is s^2, s being the length of any side of the outer triangle. This scale on each of the outer sides of the larger triangle is incorporated and adopted by the inner parallelogram encasing the diagonal line going from the corner to the opposite side of the outer triangle. The similar triangles within the parallelogram fit within the parallelogram as part of the similar triangles filling the outer triangle, and the count of these similar triangles are the same count as the similar triangles which fill each of the two triangles that make up the parallelogram, the diagonal having divided the parallelogram into two triangles. This is because each and any triangle can be treated as an equiangular, equilateral triangle. And to bring the count of the inner similar triangles from the 60º coordinate system to the 90º coordinate system is to multiply the area s^2 of the triangle by (√3)/4 which is irrational. The way of Buckminister Fuller is to measure in pure number.

The Convex Functions

I was very excited when I found this video on Non-Linear Functions on YouTube. It starts out with the Convex Functions which are defined as follows:



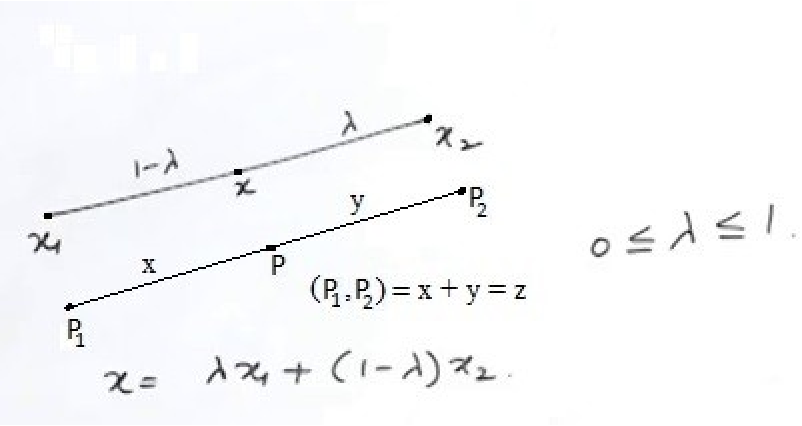
Convex Set

A set $S \subseteq R^n$ is called convex if the line segment joining any two points of S is in S. Mathematically,
for all $x_1, x_2 \in R^n$ and $\lambda \in [0, 1]$,

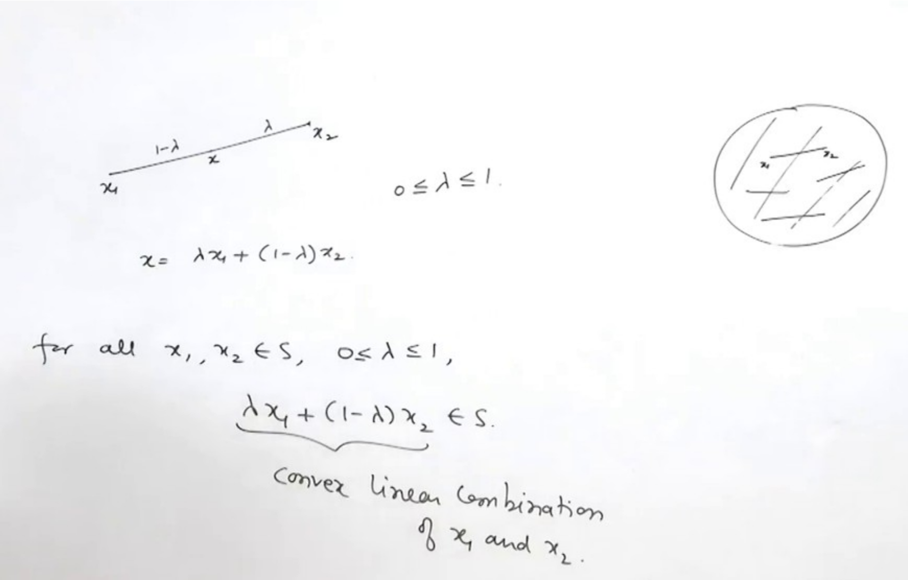
$$\lambda x_1 + (1 - \lambda)x_2 \in S.$$

Examples

You can see that the line segment in the set on the right is not included in S. Now notice the treatment of the line segment (x_1, x_2).



There is a parallel between the line segment (x_1, x_2) and the segment (P_1, P_2) which is the side of an equilateral triangle. We know that x and y are the coordinates of point P. It can therefore be conjectured that (1 – λ) and λ are the coordinates of point x and λ + (1 – λ) = z. We can then let x_1 and x_2 be the multipliers, such that $x = \lambda x_1 + (1 - \lambda) x_2$ is the equation of the line x_3.



$0 \le \lambda \le 1.$

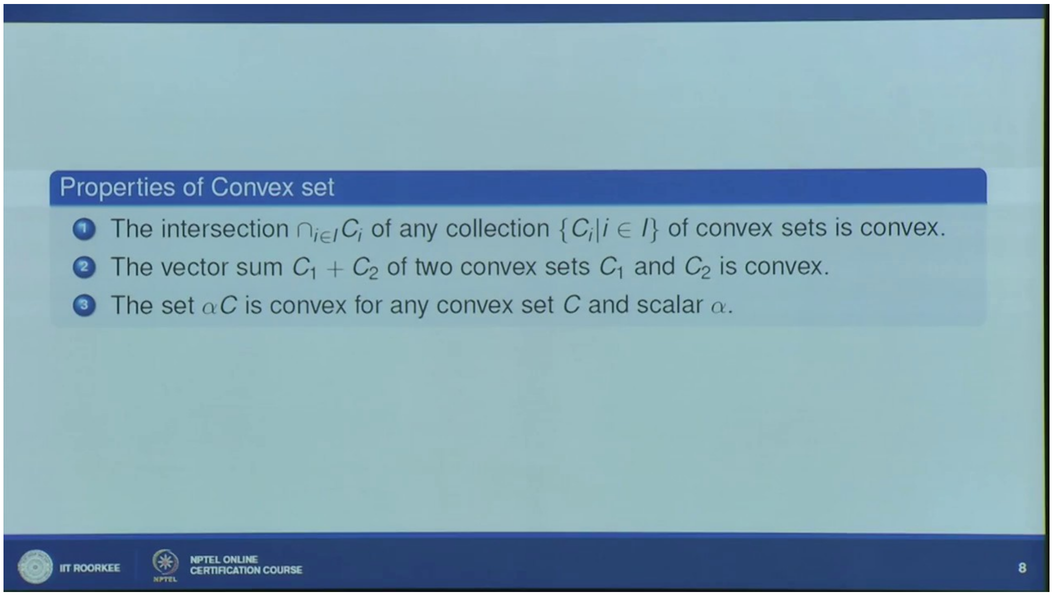
$x = \lambda x_1 + (1-\lambda) x_2.$

for all $x_1, x_2 \in S$, $0 \le \lambda \le 1$,

$$\underbrace{\lambda x_1 + (1-\lambda) x_2}_{\text{convex linear combination of } x_1 \text{ and } x_2.} \in S.$$

I can add that if $\lambda x_1 + (1 - \lambda) x_2$ ε S and the line segment x_3 is one side of an equilateral triangle, then S is the set of all equilateral triangles. Also, if x_1, x_2 ε S, then x is also in S. Therefore, any point P(λ, (1 – λ)) (where λ = x and (1 – λ) = y, the coordinates of P) on any side of any equilateral triangle is also in S.

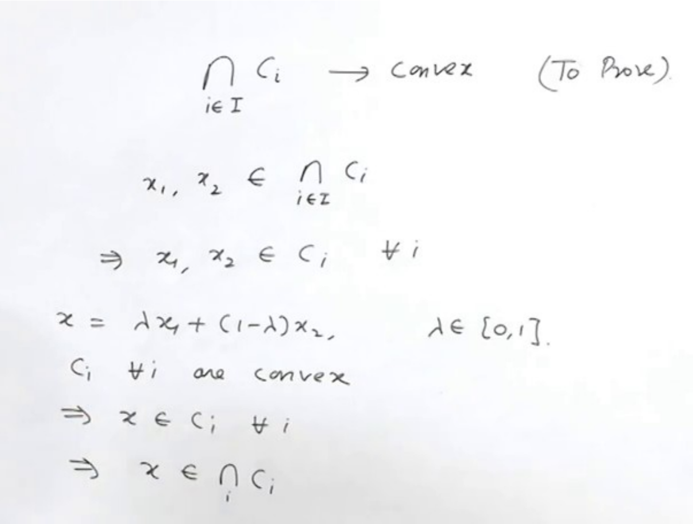
There are several properties of convex sets.



Properties of Convex set

1. The intersection $\cap_{i \in I} C_i$ of any collection $\{C_i | i \in I\}$ of convex sets is convex.
2. The vector sum $C_1 + C_2$ of two convex sets C_1 and C_2 is convex.
3. The set αC is convex for any convex set C and scalar α.

The proofs of these properties follow:



$\bigcap_{i\in I} C_i \rightarrow$ convex (To Prove)

$x_1, x_2 \in \bigcap_{i\in I} C_i$

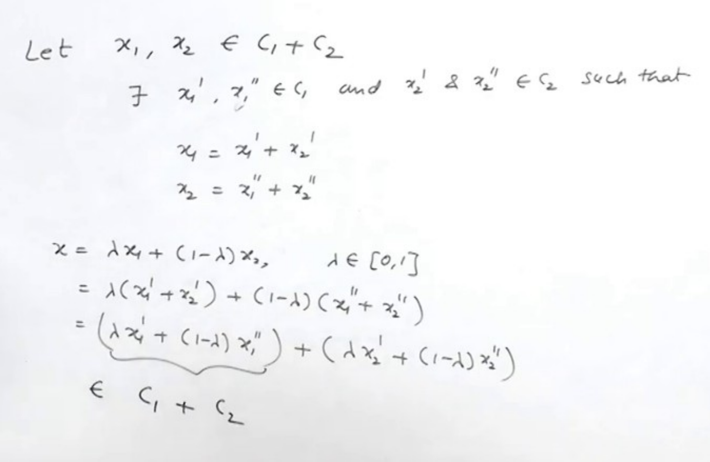
$\Rightarrow x_1, x_2 \in C_i \quad \forall i$

$x = \lambda x_1 + (1-\lambda)x_2, \quad \lambda \in [0,1].$

$C_i \ \forall i$ are convex

$\Rightarrow x \in C_i \quad \forall i$

$\Rightarrow x \in \bigcap_i C_i$



Let $x_1, x_2 \in C_1 + C_2$

$\exists \ x_1', x_1'' \in C_1$ and x_2' & $x_2'' \in C_2$ such that

$x_1 = x_1' + x_2'$

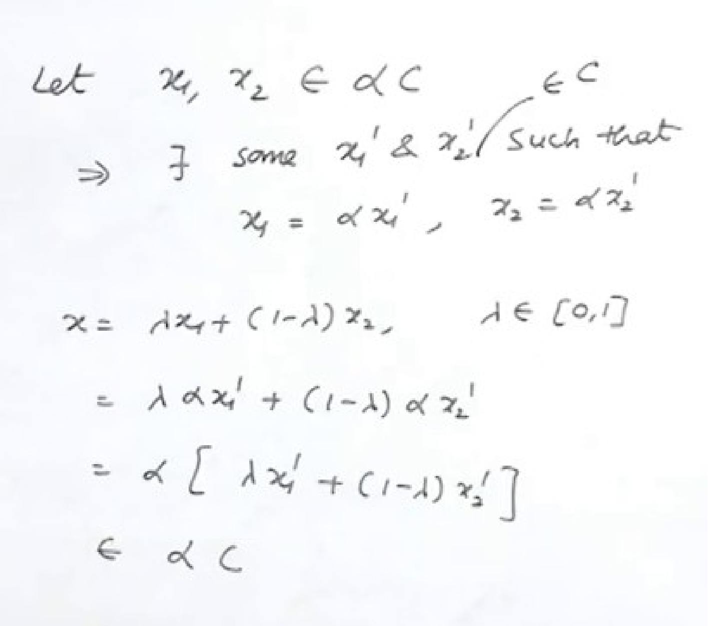
$x_2 = x_1'' + x_2''$

$x = \lambda x_1 + (1-\lambda)x_2, \quad \lambda \in [0,1]$

$= \lambda(x_1' + x_2') + (1-\lambda)(x_1'' + x_2'')$

$= \underbrace{(\lambda x_1' + (1-\lambda)x_1'')} + (\lambda x_2' + (1-\lambda)x_2'')$

$\in C_1 + C_2$



Let $x_1, x_2 \in \alpha C$

$\Rightarrow \exists$ some x_1' & x_2' $\in C$ such that

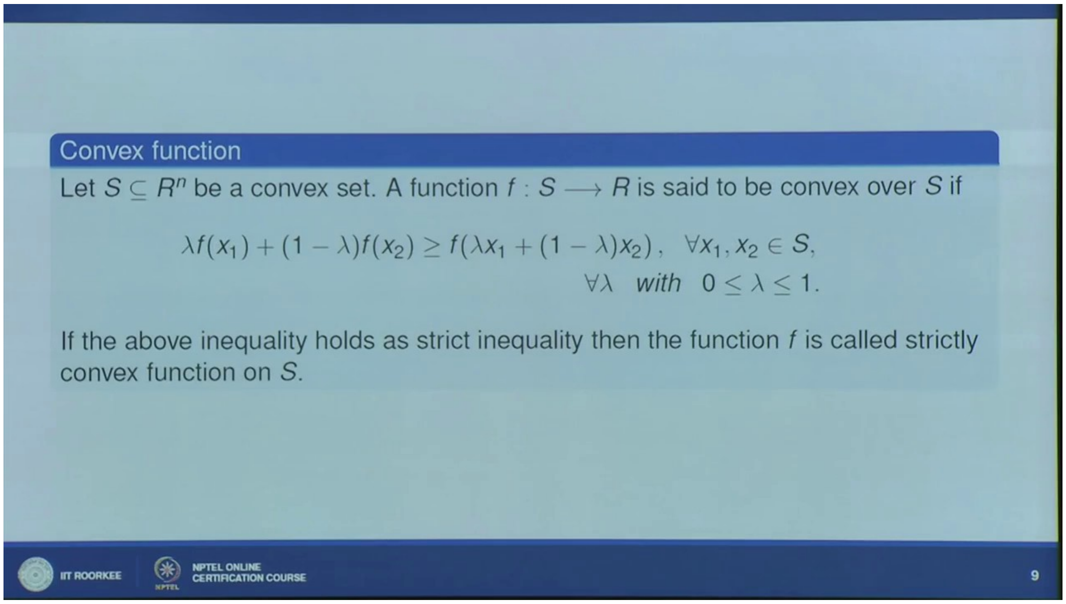
$x_1 = \alpha x_1', \quad x_2 = \alpha x_2'$

$x = \lambda x_1 + (1-\lambda)x_2, \quad \lambda \in [0,1]$

$= \lambda \alpha x_1' + (1-\lambda)\alpha x_2'$

$= \alpha\left[\lambda x_1' + (1-\lambda)x_2'\right]$

$\in \alpha C$

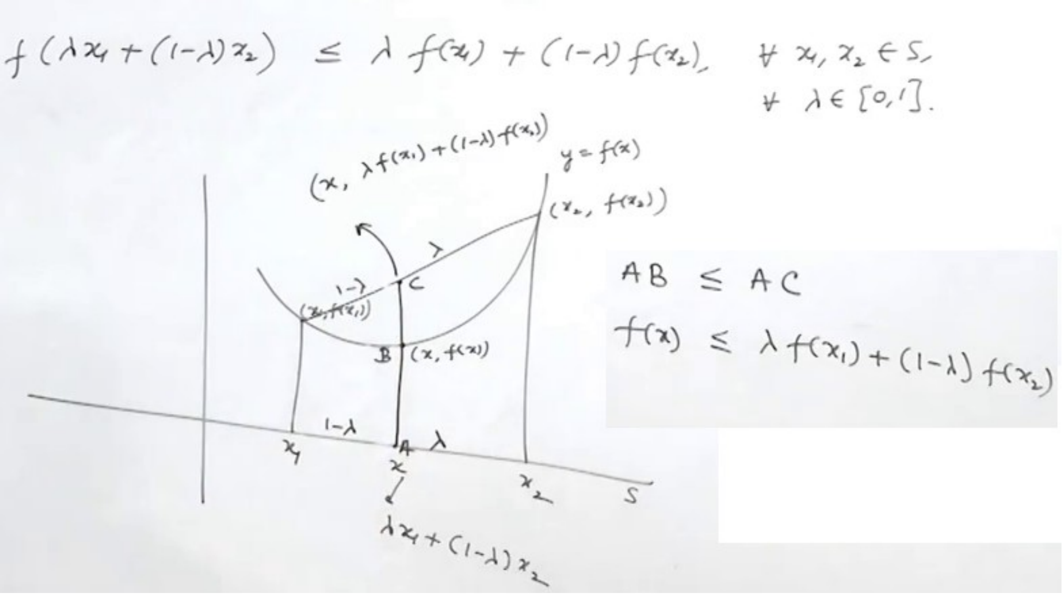


Convex function

Let $S \subseteq R^n$ be a convex set. A function $f : S \longrightarrow R$ is said to be convex over S if

$$\lambda f(x_1) + (1-\lambda) f(x_2) \geq f(\lambda x_1 + (1-\lambda)x_2), \quad \forall x_1, x_2 \in S,$$
$$\forall \lambda \quad with \quad 0 \leq \lambda \leq 1.$$

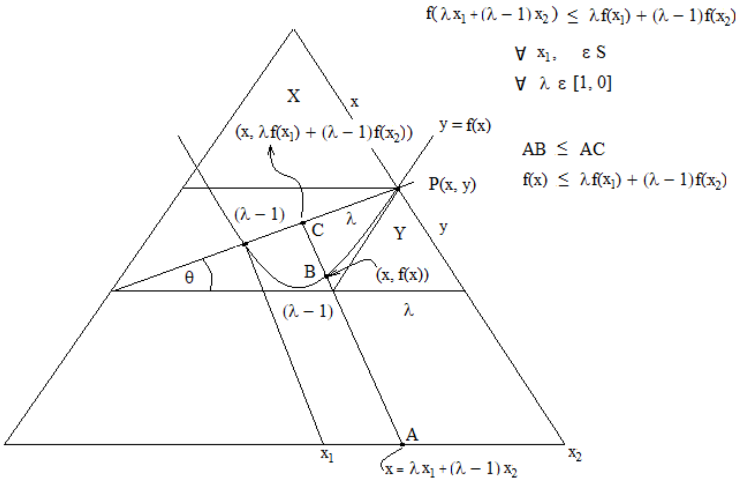
If the above inequality holds as strict inequality then the function f is called strictly convex function on S.



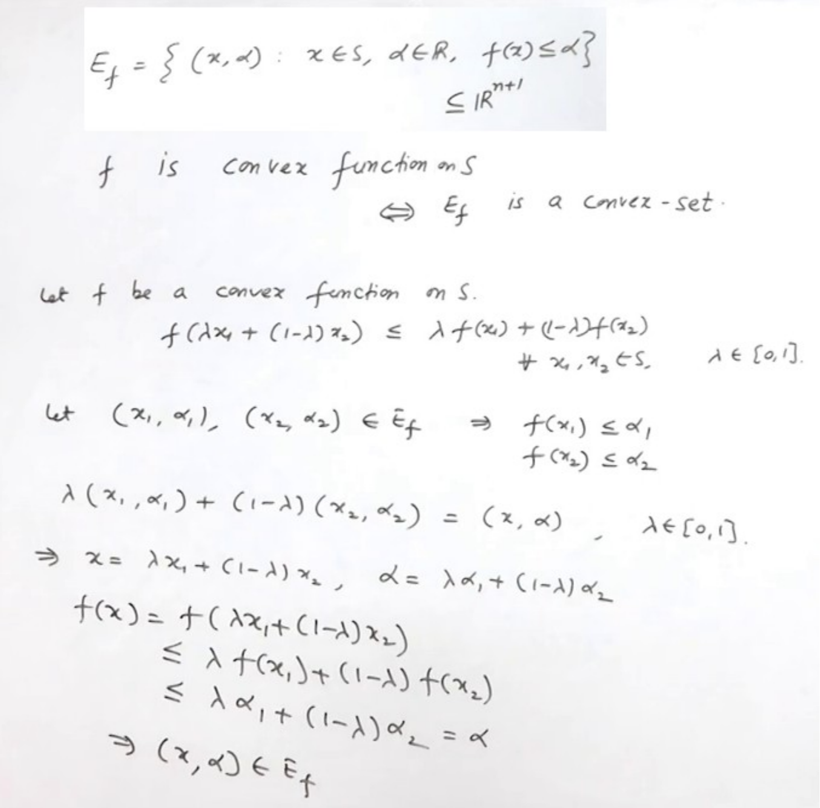
$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2), \quad \forall x_1, x_2 \in S, \quad \forall \lambda \in [0,1].$$

$$AB \leq AC$$
$$f(x) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

Convex Function within an equilateral triangle.



There is another property that comes from convex functions.



$$E_f = \{ (x, \alpha) : x \in S, \alpha \in \mathbb{R}, f(x) \le \alpha \}$$
$$\subseteq \mathbb{R}^{n+1}$$

f is convex function on S
⇔ E_f is a convex-set.

Let f be a convex function on S.

$$f(\lambda x_1 + (1-\lambda) x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2)$$
$$\forall\ x_1, x_2 \in S, \quad \lambda \in [0,1].$$

Let $(x_1, \alpha_1), (x_2, \alpha_2) \in E_f \Rightarrow f(x_1) \le \alpha_1$
$f(x_2) \le \alpha_2$

$$\lambda (x_1, \alpha_1) + (1-\lambda)(x_2, \alpha_2) = (x, \alpha), \quad \lambda \in [0,1].$$

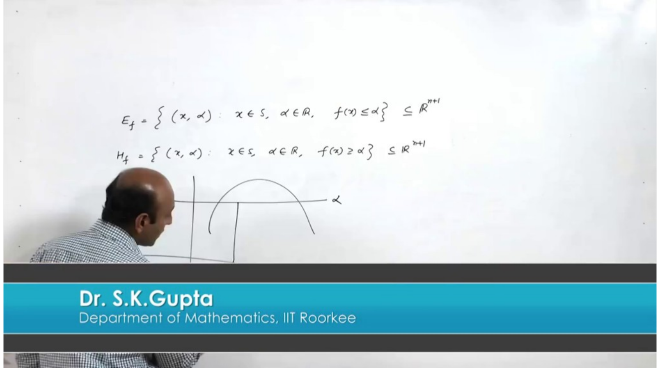
$$\Rightarrow x = \lambda x_1 + (1-\lambda) x_2, \quad \alpha = \lambda \alpha_1 + (1-\lambda) \alpha_2$$

$$f(x) = f(\lambda x_1 + (1-\lambda) x_2)$$
$$\le \lambda f(x_1) + (1-\lambda) f(x_2)$$
$$\le \lambda \alpha_1 + (1-\lambda) \alpha_2 = \alpha$$
$$\Rightarrow (x, \alpha) \in E_f$$

Notice that $\lambda(x_1, \alpha_1) + (1 - \lambda)(x_2, \alpha_2) = (x, \alpha)$, where λ ε [0, 1], implies that

$x = \lambda x_1 + (1 - \lambda)x_2$, and $\alpha = \lambda\alpha_1 + (1 - \lambda)\alpha_2$.

This shows that points x and α can be combined into a higher order (x, α) where all the points x are on one coordinate axis and all points α are on the other coordinate axis. This higher ordering is called the state system, state equations, and state plane or graph.

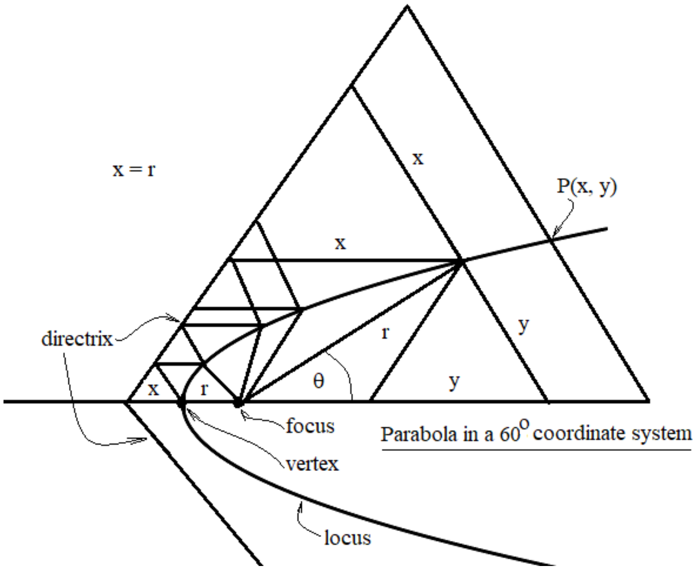


The class found on YouTube is *Convex Sets and Functions* in Nonlinear Programming from NPTEL (National Programme on Technology Enhanced Learning).

The Conic Sections

Parabola

The focus of the parabola is the set of points in which the radius r from the center of the hexagonal coordinate axes is equal to the bottom of the x triangle so that x = r, but the length of r = x cos θ + y sin θ, where x and y are the line segments that make up the outer most z axis.



The parabola in this case, cuts the z axis into the x and y coordinates and the x and y triangles. At each point on the locus of the parabola, we have the corner of an equilateral triangle in which the x and z axes come together. The directrix is the y axis above the x axis and the – z axis below the x axis unless the focus or the vertex is the center point of the hexagonal graph. Otherwise, let the vertex and focus be on the positive x axis.

When x = r, x + r = k, a constant.

The point at the focus is $P_0(x_0, 0)$, and the end point of r is $P_1(x, y)$, so that the distance between these two points is $(x - x_0) + (y - 0)$, and since x_0 is usually designated as p, we have $r = (x - p) + y$ when $\theta = 0$. Then the distance between point $P_2(0, y)$ at the directrix and the point $P_1(x, y)$ on the locus is $(x - 0) + (y - y)$ or is simply x.

Since $x = r$, then the next $x = (x - p) + y$, but with the continually expanding or shrinking x triangles, the next $(x - p) = x \cos \theta$, and the next $y = y \sin \theta$, so the next $x = x \cos \theta + y \sin \theta$.

If the point on the locus is $P_0(x, y)$, then the next point is $P_1(x \cos \theta, y \sin \theta)$. As the angle θ changes, x and r increase or decrease at a constant rate as $k = x + r$.

Important discoveries

Note: We have some really important discoveries pertaining to the algebra of a 60° coordinate system.

1. The diagonal of a square is unity.
2. The death of π : The measure of circular length is based on 3 and 6.
3. The death of the Pythagorean Theorem: $x + y = z$ is the distance equation instead of $z = \surd(x^2 + y^2)$.
4. The triangular root of an area is a line such as $x = \surd(x^2)$.
5. The tetrahedral roots of a number n^3 is the plane of an equilateral triangle.
6. $r = x^2 + xy + y^2$ (not $x^2 + 2xy + y^2$) is the length of a line r from a corner of an equilateral triangle to its opposite side.
   a. A parallelogram is treated as a rectangle xy when its internal angles are 60° and 120°.
   b. The parallelogram is divided into two triangles whose inner similar triangles are the same number as the equilateral triangles found in their place before the division into similar triangles.
7. $Nx^2 = V^2 \Rightarrow Nx = V$ (if x^2 is a factor of V^2).

Note: $y_n^2 = y^2/(2n + 1)$, $y_n^2[(n + 1)/(2n + 1) + n/(2n + 1)] = y^2 = \Delta x^2$; let F be called a Fourier, so $F = \Delta x^2 - y_n^2(n/(2n + 1))$

Note: all the triangles within the major triangle are counted using triangular numbers. This leads to being an analog for harmonics of a string. This leads to the orthogonality of sin and cosine functions. This leads to the basis vectors of the 60° coordinate system.

The Three Spaces

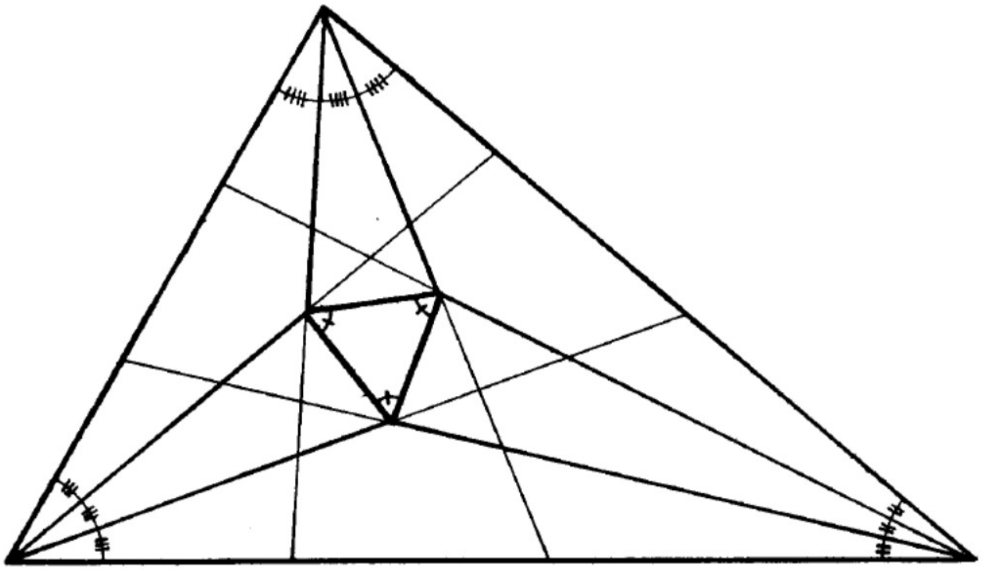
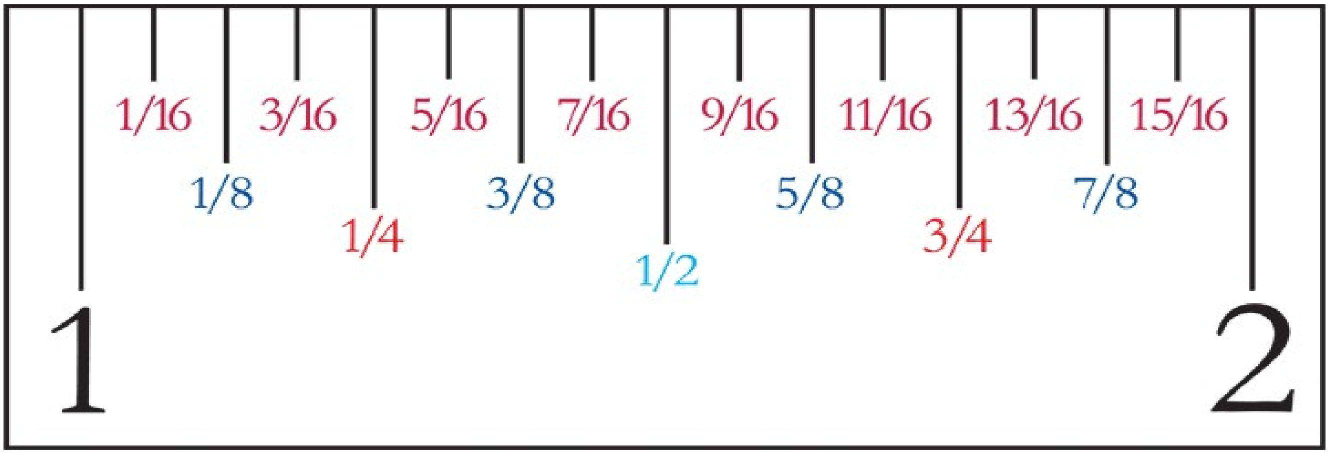


Fig. *100.51 Morley Theorem:* The trisectors of any triangle's three angles describe an equiangular triangle.

1. linear space
2. variable linear space
3. nonlinear space

The Fractal Nature of the Number Line



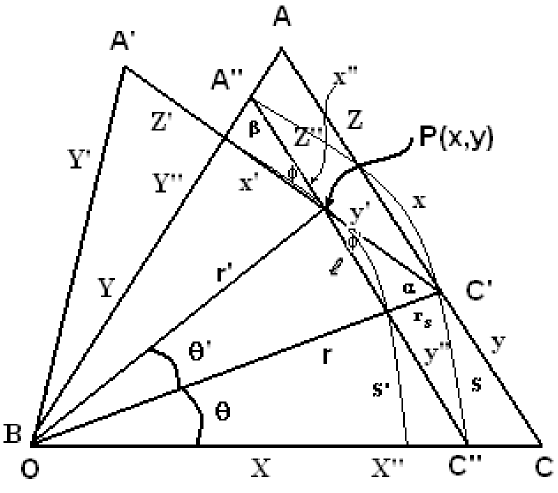
A lot has been said of the continuity of the natural numbers or the number line, sometimes called E_1. It has been said that within any interval (a, b) on E_1 that there is an infinite series S of divisions 1/n, such that a < 1/n < b, as n approaches infinity. But what is between each division? For example, within 1/aaaa … m < 1/aaaa … q < 1/aaaa … n, there is always a q such that m < q < n, meaning that all the digets aaaa … are the same and only m, q, and n are different and q is between m and n on E_1 .

The Length of r

Any line r has a length of x + y. Within an equilateral triangle ABC with sides X, Y, and Z, draw a line **r** from O to the opposite side Z. Line **r** is the base of another equilateral triangle A'BC' and is also a radius of arc s. Swing a copy of **r** down to X so that **r** coincides with X. From the endpoint of **r**, draw a line ℓ parallel to Z up to Y. Line ℓ completes another equilateral triangle A"BC". The triangle A'BC' therefor is only the triangle A"BC" rotated at an angle of θ.

Therefore, Z' = Z", **r** = X", and Y' = Y" by definition. Draw a line **r'** from O to Z' where triangles A'BC' and A"BC" intersect at point P(x,y). Because Z' crosses Z", produce angles ϕ and ϕ', the angles ϕ and ϕ' are equal.

Radius r' divides Z' into x' and y' and Z" into x" and y".



All angles of equilateral triangles are equal, so α and β are equal by definition.

Since $Z' = Z''$, $y' / x' = x''/ y''$.

The distance between ℓ and Z is $\mathbf{r_s}$. The distance between the two arcs s and s' of which both extend from X'' to Y'' is also $\mathbf{r_s}$.

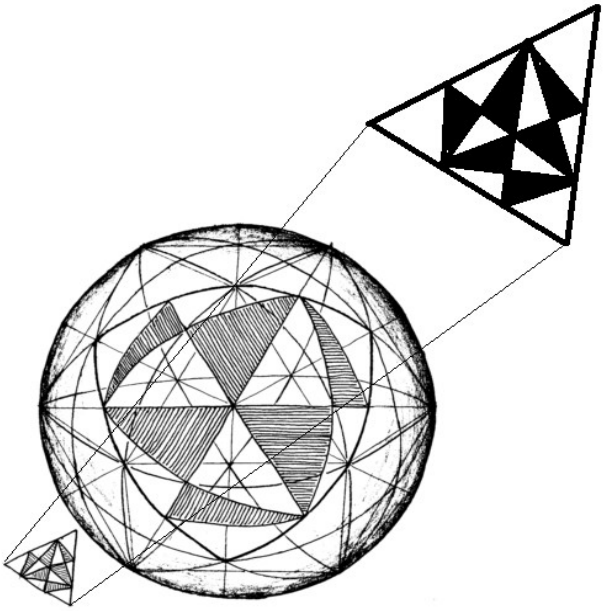
Angle α is made up of sides y' and $\mathbf{r_s}$ and angle β is made up of sides x'' and $\mathbf{r_s}$.

Because $y' / x' = x''/ y''$ and $\alpha = \beta$ and $\phi = \phi'$, then the triangles A'A''P(x,y) and C 'C '' P(x,y) are congruent.

The triangles A'A''P(x,y) and C 'C '' P(x,y) are both shortened by sides $\mathbf{r_s}$. We can call the side opposite ϕ, $\mathbf{y_s}$.

Because $\alpha = \beta$ and $\phi = \phi'$ and $\mathbf{y_s} = \mathbf{r_s}$, then $y' = x''$.
If $y' = x''$ and $Z' = Z''$, then $x' = y''$.
By definition, $Z'' = X'' = Z' = \mathbf{r}$.
Therefore, $\mathbf{r} = x' + y' = x'' + y''$.

Important Ratios



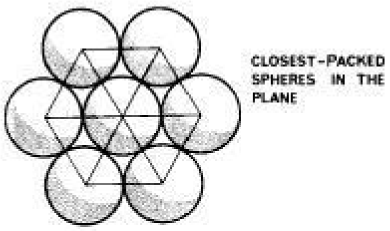
There is a ratio involved in the volume of a sphere. It is 15/3. It comes from (120/8)/(24/8), 8 because of the spherical octahedron where there are 8 faces. The spherical right triangle within the spherical equilateral triangle is 1/120th of the surface area. In both the spherical icosahedron and the spherical octahedron, there are 15 A and B quanta modules in one of the spherical triangular faces.

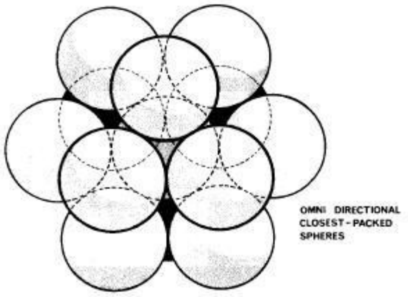
Another ratio is 20/4 related to the spherical cuboctahedron made up of 60 A and B Quanta Modules. This also has to do with the volume of a sphere. (15/3) x 4 = 60/12 = (20/4) x 3. There are 4 planes in the Vector Equilibrium and three axes in each of the 4. The Vector Equilibrium is the key to the reason why the unit sphere is 5. The cuboctahedron has 60 A and B Quanta Modules. 60/12 = 5.

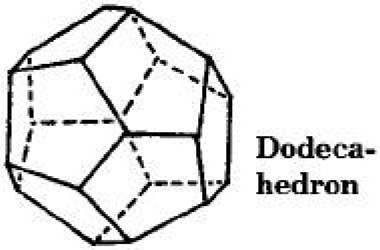
There is a relationship between the Icosahedron, the cuboctahedron, and the Dodecahedron. The cuboctahedron and the Icosahedron have the same number of vertexes, where the closest packing of spheres have their centers. Taking out the central sphere from the cuboctahedron, it contracts to a more symmetrical configuration, the Icosahedron, but the number of vertexes, which is 12, remain constant. The Dodecahedron has 20 vertexes, but it has 12 faces. The spherical Dodecahedron thus has 12 equally spaced centers on its surface where the vertexes of the encased Icosahedron touches the surface, as does the cuboctahedron. So the relationship of each of these spheres is this 12 equally spaced points. This obviously comes from the way 12 spheres pack closely around a central sphere. Thus common denominator of 12.

Formation of the Dodecahedron

Take 6 circles surrounding one circle and push them onto the center circle with equal force**.** The center circle becomes a hexagon. Beehives are the result of the most economical use of circular space. The Dodecahedron represents the most economical use of three dimensional space, and like the cube is an all–space filler. If you take 12 soft spheres surrounding a central soft sphere and each outer sphere is pushed with the same force towards the central sphere, a Dodecahedron results.







The Binomial

The **binomial theorem** states that

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n,$$

where $\binom{n}{n-\mathrm{x}}$ = C(n, n – x) = (n!/(n – x)!x!),

or equivalently, $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$

But according to the definition $(x + y)^2 = x^2 + xy + y^2$, where n = 2 and x = 1, the

binomial theorem for the 60° coordinate system becomes $(a + b)^n = \sum_{n=1}^{\mathrm{i}} \sum_{\mathrm{x}=0}^{n} a^{n-x} b^{x}$.

For n = 1 and x = 0, $(a + b)^n = a$. For n = 2 and x = 1, $(a + b)^n = a^2 + ab + b^2$.

We talked about taking a line and triangling it to make it into an area, let's now talk about the binomial $(x + y)^2$ and degrading it into a line.

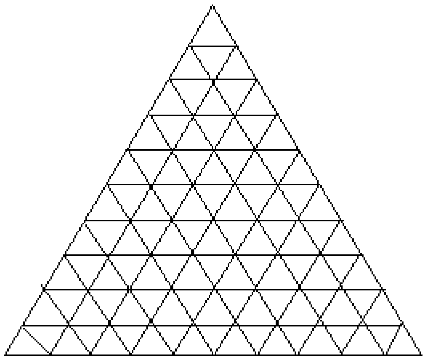
Squaring the Area of a Triangle

The area of an equilateral triangle whose side is x is comparable to the area x^2 (called “x square”) of a square whose side is x. A square can be divided into x^2 smaller squares.

An equilateral triangle is divided into smaller equilateral triangles such that the count starts with a base case of $k_o = 0$. The area of the first row is $(x = 1, n = 0)$. $k_x = 2n + 1$, where $n = 0, 1, 2, 3, \dots$, and $x = 1, 2, 3, \dots , x$ being the number of rows counted as well as a coordinate of P on Z. If k is the area of each row or a triangular number, then the next count is ($x = 2$)

$$t_x = k_x + k_{x-1},$$

where t_x is the sum of all the areas counted so far.



If you enumerate all the t_x's, you will find that $t_x = x^2$ (called “x triangled”).

n	0	1	2	3	4	5	6	7	8	9
x	1	2	3	4	5	6	7	8	9	10
t_x	1	4	9	16	25	36	49	56	81	100

Therefore, there is a one-to-one correspondence of an equilateral triangle to a square.

The coordinates of P on Z now become $\surd(x^2)$, recognizing that taking the triangular root of a number representing an area of an equilateral triangle becomes a line segment.

Note on Rotation:

Creating a rotation matrix in NumPy
The two dimensional rotation matrix which rotates points in the xy plane anti-clockwise through an angle θ about the origin is

$R=(\cos\theta\sin\theta-\sin\theta\cos\theta)$.
To create a rotation matrix as a NumPy array for $\theta=30^\circ$, it is simplest to initialize it with as follows:

```
In [x]: theta = np.radians(30)
In [x]: c, s = np.cos(theta), np.sin(theta)
In [x]: R = np.array(((c, -s), (s, c)))
Out[x]: print(R)
```

```
[[ 0.8660254 -0.5      ]
 [ 0.5        0.8660254]]
```

As of NumPy version 1.17 there is still a matrix subclass, which offers a Matlab-like syntax for manipulating matrices, but its use is no longer encouraged and (with luck) it will be removed in future.

Derivation 1

Since rotations are linear transformations, the effect of rotating a vector from the origin to some arbitrary point, P=(x,y), can be established by considering the rotation of the basis vectors $e^x \equiv (1,0)$ and $e^y \equiv (0,1)$. In the figure below, a rotation by θ takes

$e^x e^y \rightarrow e^{\prime x} = \cos\theta e^x + \sin\theta e^y, \rightarrow e^{\prime y} = -\sin\theta e^x + \cos\theta e^y$.

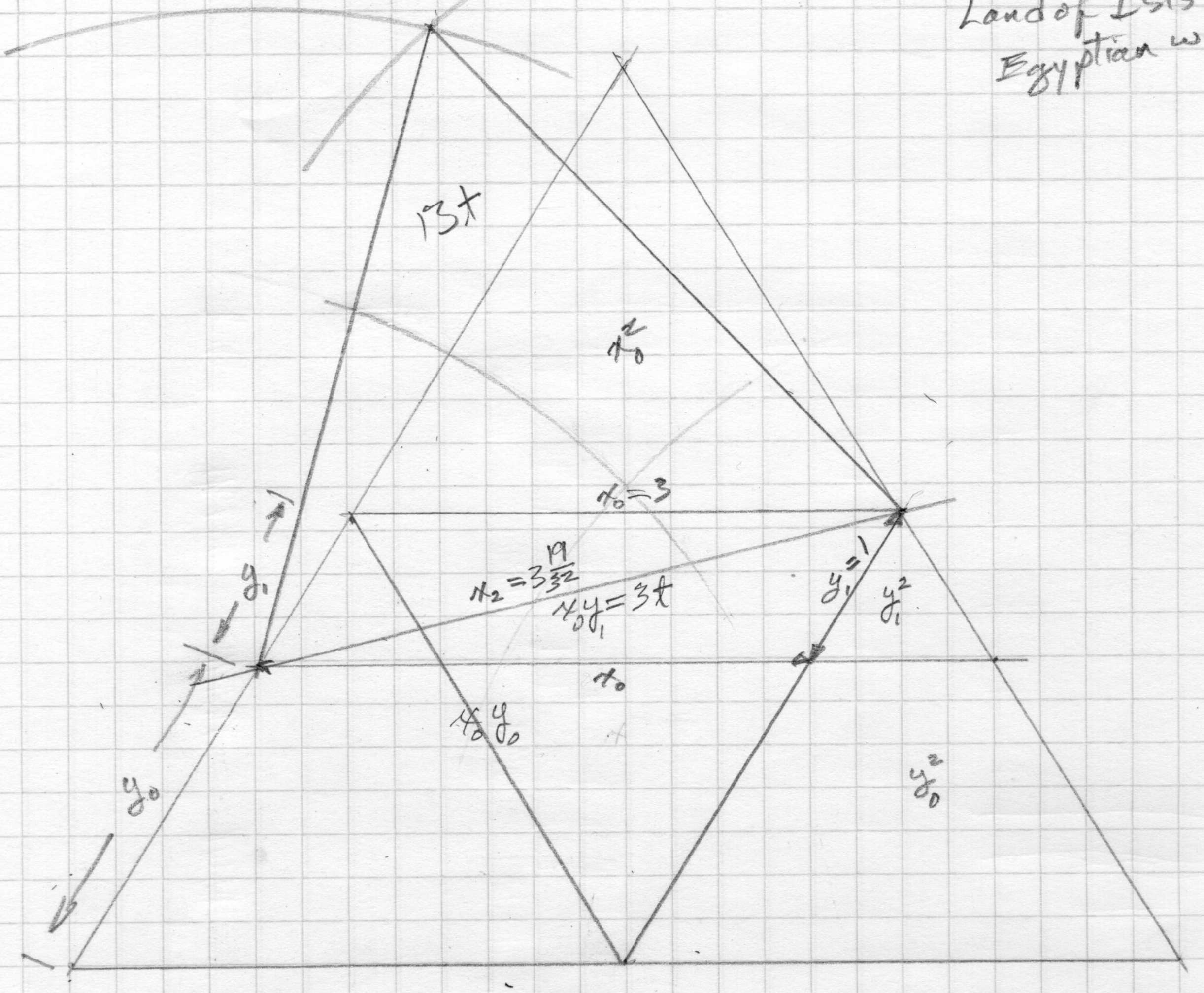
Our point P is therefore transformed from (x,y)≡xe^x+ye^y to:

$P' = xe^{\prime x} + ye^{\prime y} = (x\cos\theta - y\sin\theta)e^x + (x\sin\theta + y\sin\theta)e^y$.

That is,

$P' = RP = (\cos\theta\sin\theta - \sin\theta\cos\theta)(xy)$.

Tennessee =
Land of Isis
Egyptian word



$\sqrt{3^2 + 3{,}1 + 1^2} = \sqrt{13} = 3{,}6055$

✓ $= 3\frac{19}{32}$

$\sqrt{3^2 + 1^2} = \sqrt{10} = 3.162 = 3\frac{5}{32}$

$\left(3\frac{19}{32}\right)^2 = 13$ triangles

$$x_2 = \sqrt{x_0^2 + x_0 y_1 + y_0^2}$$

area of equilateral triangle $= x^2 + xy + y^2 = (x+y)^2 = z^2$